
A thesis submitted toward the degree of
Master of Science in Electrical and Electronic Engineering

by

Lior Gazit

September 2016

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This research was carried out in the Department of Electrical Engineering – Systems, under the supervision of Prof. Hagit Messer-Yaron

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Abstract

An innovative method published in 2006 by Prof. Messer-Yaron (Messer et al., 2006) introduced the use of commercial cellular networks for environmental monitoring of precipitation by exploiting existing microwave-transmission data. This method for precipitation mapping is revolutionary in its approach. Current weather monitoring techniques lack the ability to capture precipitation maps with sufficient spatial resolution. Since its publishing, many studies took place, both under the supervision of Prof. Messer-Yaron, and in various institutions around the world. They revolved around numerous aspects of the subject such as geophysical considerations, signal and data processing approaches, classification of the precipitation phenomena, etc.

The purpose of the research presented here is to form criteria for successful reconstruction of a precipitation map. These criteria, novel in its application to microwave precipitation monitoring, take into account a given microwave measurements system and its data, and calculate the probability of achieving proper estimation. In other words, given a file of microwave measurements, one would apply the criteria and will immediately know the expected probability of reconstruction for the rainfall map. Moreover, the suggested method also serves as a guide for system synthesis, telling what adjustments may allow a higher probability of reconstruction. The methods applied are borrowed from the world of image-processing, linear-operators, and statistical-modeling.

The scientific contribution is three fold. First, this work provides an answer to the above question, thus completing a meaningful part of the puzzle of the monitoring system whose significance to science and industry is without a question. Second, it marks another contribution to the study of linear systems of equations and their application is real-world measurement problems. More specifically, it presents a unique method of image-sampling where the image is observed by random straight lines. Third, this work presents a statistical-study of microwave-links, a study which is applied to the statistical-modeling of such links, allowing large-scale simulations.
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<tr>
<td>$a', a, b', b, c, c_n, d, y, y_l$</td>
<td>Scalars.</td>
</tr>
<tr>
<td>$X(\cdot), R(\cdot), \Lambda(\cdot)$</td>
<td>Scalar-valued functions.</td>
</tr>
<tr>
<td>$r, s_i$</td>
<td>Vectors.</td>
</tr>
<tr>
<td>$A$</td>
<td>Microwave attenuation, [dB/Km].</td>
</tr>
<tr>
<td>$R$</td>
<td>Rain-rate, [mm/hr].</td>
</tr>
<tr>
<td>$X$</td>
<td>A matrix representing the attenuation-field.</td>
</tr>
<tr>
<td>$[X]_{i,j}$</td>
<td>The element from the $i$–th row and $j$–th columns of matrix $X$.</td>
</tr>
<tr>
<td>$x$</td>
<td>A column vector, the column-representation (reshape) of matrix $X$. $x = CR{X}$.</td>
</tr>
<tr>
<td>$x_s$</td>
<td>A sparse representation of vector $x$. Achieved by a sparsifying linear transformation.</td>
</tr>
<tr>
<td>$\Lambda_l$</td>
<td>A matrix representing the $l$–th link in the plane. It has zeros almost everywhere, and positive components where the link lays.</td>
</tr>
<tr>
<td>$\lambda_l$</td>
<td>A column vector, the column-representation (reshape) of matrix $\Lambda_l$. $\lambda_l = CR{\Lambda_l}$.</td>
</tr>
<tr>
<td>$A$</td>
<td>A matrix holding all the links. $\lambda_l$ is the $l$–th column of $A$.</td>
</tr>
<tr>
<td>$N$</td>
<td>The number of pixels in the region of interest, also is the number of unknown components in the attenuation-field $X$.</td>
</tr>
<tr>
<td>$n$</td>
<td>The number of microwave-links available for observations.</td>
</tr>
<tr>
<td>$k$</td>
<td>The number of non-zeros elements in vector $x_s$, i.e. $k = |x_s|_0$.</td>
</tr>
<tr>
<td>$\delta$</td>
<td>The compression ratio $\frac{n}{N}$.</td>
</tr>
<tr>
<td>$\rho$</td>
<td>The complexity ratio $\frac{k}{n}$.</td>
</tr>
</tbody>
</table>
| $s$ | The sparsity $\frac{k}{N}$. Notice that the higher $s$ is, the
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<thead>
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<th>Symbol</th>
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<tr>
<td>S</td>
<td>A matrix representing a sparsifying transform, i.e. ( x_s = Sx ).</td>
</tr>
<tr>
<td>M</td>
<td>A matrix obeying ( M = A^T S^{-1} ), its significance is made clear in the report.</td>
</tr>
<tr>
<td>( |x|_p, p &gt; 0 )</td>
<td>The ( p )-norm, ( l_p ).</td>
</tr>
<tr>
<td>( |x|_0 )</td>
<td>The number of non-zeros elements in vector ( x ). This can be derived by calculating the ( l_p ) as ( p \to 0 ).</td>
</tr>
<tr>
<td>( P(\cdot) )</td>
<td>Probability of an event.</td>
</tr>
<tr>
<td>( 1_N )</td>
<td>A row vector of length ( N ) filled with 1's.</td>
</tr>
<tr>
<td>( 0_{N\times N} )</td>
<td>An ( N \times N ) matrix full of 0's.</td>
</tr>
<tr>
<td>( B^T )</td>
<td>The transpose of matrix ( B ).</td>
</tr>
<tr>
<td>( CR{\cdot} )</td>
<td>Matrices reshape operator, mapping a matrix to a column vector, ( CR{\cdot} : \mathbb{R}^{i \times j} \to \mathbb{R}^{i \times j \times 1} ).</td>
</tr>
<tr>
<td>Haar{\cdot}</td>
<td>The 2-D Haar transform.</td>
</tr>
<tr>
<td>( D_x{\cdot} )</td>
<td>An operator representing the horizontal differentiation of a matrix (column differences). ( D_x{\cdot} : \mathbb{R}^{i \times j} \to \mathbb{R}^{i \times j} ).</td>
</tr>
<tr>
<td>( D_y{\cdot} )</td>
<td>An operator representing the vertical differentiation of a matrix (row differences). ( D_y{\cdot} : \mathbb{R}^{i \times j} \to \mathbb{R}^{i \times j} ).</td>
</tr>
<tr>
<td>Standard Normal Distribution</td>
<td>Normal distribution with mean 0, and STD 1.</td>
</tr>
<tr>
<td>RSL</td>
<td>Received Signal Level</td>
</tr>
<tr>
<td>dB</td>
<td>Decibel</td>
</tr>
<tr>
<td>DFT</td>
<td>Discrete Fourier Transform</td>
</tr>
<tr>
<td>SoS</td>
<td>Sum of Squares</td>
</tr>
<tr>
<td>SMHI</td>
<td>Swedish Meteorological and Hydrological Institute</td>
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1 Introduction

This work addresses the matter of precipitation-monitoring in general, and rain-mapping in particular. More specifically, it tends to the need of a high resolution real-time rain-mapping by utilizing cellular backhaul network infrastructure, and by taking advantage of the physical interaction between rain and microwave signals. This interaction is scientifically defined and is already observed and utilized.

1.1 Motivation

As wireless cellular networks are spread over populated regions, they present a unique opportunity. Cellular providers, as a part of their operational routine, monitor microwave signal power propagation along the wireless backhaul channels. It is the transmitted microwave signal that carries information about the propagation path. The work in (Giuli et al., 1991; Giuli et al., 1999) describes how a microwave channel, designed for this cause, can be applied to study meteorological phenomena. Capitalizing on that and suggesting an opportunistic approach, (Messer et al., 2006) describes the commercial backhaul network for the same need. This novel approach makes the research more practical and applicable, especially from a cost perspective.

The work in (Messer et al., 2006) was followed up with more work exploring methods in signal-processing for microwave signals and for deriving estimations of the rain-field (Leijnse et al. 2007; Leijnse et al. 2008; Corke et al., 2010; Overeem et al. 2011; Cherkassky et al. 2012; Harel and Messer, 2013; Liberman and Messer, 2014; Ostrometzky et al. 2015; David et al. 2016).

As the notion of deriving the rain-field from random backhaul cellular signal observations becomes clear, a new need presents itself. How well do these methods estimate the rain-field? They yield a tentative rain-field, but without a ground-truth field to compare to, how would we know how precise the reconstruction really is? The work here is concerned with this question and presents an answer in the form of mathematical criteria that tells whether good reconstruction is feasible based on a given set of microwave measurements.

1.2 Background to Precipitation Monitoring

As a preliminary step, let’s observe the more common approaches to monitoring rainfall. There are three common methods for monitoring precipitation (Strangeways, 2006): rain-gauges and disdrometers, weather-radars, and weather-satellites.

1.2.1 Rain Gauges

Rain gauges are quiet simple. In its simplest form, a rain gauge is a container collecting accumulating rain and monitoring the rain-rate. Rain-gauges are cheap, easy to deploy and are easy to operate. They are considered reliable and accurate in their rainfall rate monitoring around ground level. A critical disadvantage of rain gauges is their spatial deficiency, as they present the rain phenomenon in one particular point that may, or may not, be representative of the entire rain-field.
1.2.2 Disdrometers

A disdrometer is also a local instrument. It is designed to measure but also differentiate the various precipitation phenomena. It applies a dense array of beams and corresponding sensors for classifying precipitations based on the patterns that they cause. The disdrometers’ advantage is their high precision. However, they are expensive. Moreover, just like the rain-gauges, they are completely local.

1.2.3 Weather Radars

The weather radar, being located in one position but observing its entire surrounding, is capable of yielding a macro view of the rain-field. This is its clear advantage on the previous two. Since the earth’s surface is curved, the farther a point is from the radar, the lower the ground is. Thus the radar has this as a limitation to capturing ground-level precipitation. The weather radar is quite costly.

1.2.4 Weather Satellites

Weather satellites observe weather phenomena from outer space. They are equipped with various sensing instruments like cameras, and radar. Thus, a clear disadvantage is their ability to capture ground-level precipitation as all layers of the atmosphere are being observed simultaneously.

1.3 Introducing the Microwave Link

The weather radar and satellite utilize the interaction between the presence of moisture in the air and the effect it has on electromagnetic waves. Figure 1.1 illustrates a microwave-link. The idea of setting up a wireless microwave channel for precipitation monitoring was first presented in (Giuli et al., 1991). Each link provides an aggregated observation of the precipitation in its path, an ensemble of such links can be integrated to provide a map of precipitation in the entire region.
Figure 1.1 - Illustrating a single microwave-link.

Figure 1.2 presents attenuation levels for waves in the cellular spectra in the presence of different rain-rates (Ulaby et al. 1981).

While the cellular providers’ initiative was to make up for the attenuation phenomena by developing technology to minimize its effect, their instruments and methods allow not only a compensation for it, but to also observe and document these effects. In particular, these instruments document the instantaneous Received Signal Level (RSL) for each microwave link.
Observing RSL measurements is the key advantage that allows for spatial precipitation monitoring (Messer et al., 2006). Messer et al.’s work presented both the potential hidden in microwave-links RSL data, and the operational opportunity presented by the fact that these data are freely available to all cellular providers as a “by-product” of their operations. Following that publishing, various research groups recognized the potential and started their own studies, such as (Rahimi and Upton, 2006; Berne and Uijlenhoet, 2007; Upton et al., 2007; Leijnse et al., 2007a). In (Goldshtein et al., 2009) the authors developed a method for estimating a two-dimensional rainfall intensity field based on multiple path-integrated RSL observations. In (Heder and Bertok, 2009) a method for detection of RSL attenuation in single link caused by sleet events was proposed.

In general, the signal attenuation that drives the RSL may be caused by various atmospheric phenomena. In particular, rain, sleet, snow, and fog. As this work focuses on the interaction between rain and RSL, let’s observe the mathematical model relating two. The formula relating electromagnetic attenuation and rain-rate is simply called the power-law (Olsen et al., 1987). It is an approximation, and is considered valid for convective rains and signals in mid-range frequencies (i.e. over 1 [GHz] and below optical). The formula in (1.3.1) presents the detailed formula for the relationship between the two:

$$A = a'R^{b'} \left[ 1 + \sum_{n=2}^{\infty} c_n f^n R^n d \right]$$

(1.3.1)

where $a', b'$, and the $c_n$ coefficients are constants depending on the frequency, rain-temperature, and drop size distribution. $f$ is the wave frequency, and $d$ is a constant depending on the parameters of the drop size distribution. $A$ represents the attenuation and is in units of $[dB/Km]$ and, $R$ is in $[mm/hr]$, representing the rain-rate. In (Olsen et al., 1987) the authors showed that the following approximation:

$$A = aR^b$$

(1.3.2)

is appropriate and provides adequate results. The validity of the approximation is derived by experimental results. Moreover, often the power coefficient $b$ is approximated to be 1, thus making the relationship linear. Such approximation suits wavelengths at about 1 cm. When discussing dedicated microwave links, Giuli et al. (Giuli et al., 1991; Giuli et al., 1999) choose wave frequencies which suit the linear A-R relationship. In such case the power-law is reduced to a simple proportion:

$$A = aR$$

(1.3.3)

The power-law presented above provides a relationship for every specific point in space. For a constant rain-field (i.e. constant along a link’s path) we could simply set the observed RSL in the place of $A$, and immediately find the $R$ that corresponds to the rain-field around that link. But, if the rain-field is space-dependent, i.e. not constant in space but vary along the path of the link, a more complex calculation would have to be introduced. Chapter 3, introducing the mathematical model for this work, dives deeper into this scenario.
1.4 Sensors Network

Let’s regard every microwave-link as a sensor. Individually, every one of these sensors may of low quality with regards to its ability to yield a mapping of the rain-field. Not only is it an integration of all the phenomena along its path, but the cellular providers also alter the RSL data based their operational needs and limitations. These alterations can be divided to three categories: 1 – various quantizations of the RSL levels (Goldshtein et al., 2009), 2 – various temporal sampling rates of the RSL time-series (Messer and Sendik, 2015), 3 – various aggregations of the RSL values (e.g. taking the temporal mean, the min/max, or the instantaneous value) (Messer and Sendik, 2015). In spite of that, sophisticated signal-processing techniques can utilize the sensor-network and compensate for those degradations (Leijnse et al. 2008; Rayitsfeld et al. 2012; Liberman and Messer, 2014).

Our sensors-network consists of all wireless backhaul links in the area of interest. The RSL of each link is monitored to ensure connectivity, and in most cases the RSL measurements are transmitted to the provider’s center for logging. Thus, the cellular provider has access to measurements of many microwave-links. Figure 1.3 (Overeem et al., 2013) portrays the distribution of microwave links in the Netherlands and in Israel.
In Section 4 I explore the characteristics of microwave-links in the sensor-network. Later I utilize these characteristics for the development of mathematical criteria relating a data-set to the expected feasible reconstruction.

1.5 Mapping Rain-Fields by a Sensors-Network

The general purpose of Prof. Messer-Yaron’s research is the reconstruction of rainfall maps. A comprehensive survey of mapping rain-fields via a cellular-network can be found in (Messer and Sendik, 2015).

It should be stressed that rainfall mappings based on measurements from cellular backhaul networks was found to match radar rainfall maps (Leijnse et al., 2007b; Zinevich et al. 2008; Overeem et al., 2013), and rain-gauge measurements (Messer et al., 2008; Zinevich, et al. 2010). Thus it is clear that in areas with no radar coverage (such as parts of Africa and South America) cellular backhaul networks provide a practical alternative to special purpose and expensive weather radar networks.
Note that this method is only feasible over land. This approach is not suitable for rain mapping over oceans, where satellites are the only alternative.

It was shown that the local accuracy of rainfall achieved by a microwave link can reach a high level of accuracy with a correlation that can be as high as 0.86, in cases where the microwave-link is in a vicinity to the reference gauge (Messer et al., 2006, Liberman et al., 2014). In (Tai-Chang, 2015) the authors present a correlation of as high as 0.9647 between the rainfall extracted using their inversion method, and disdrometers.

An issue still unexplored is concerned with the achievable spatial resolution of the reconstructed rain-field. Another question to be presented for a specific set of measurements is whether reconstruction is at all feasible.

1.6 Problem Statement

My work takes a key part within Prof. Messer-Yaron’s vast research for its exploration and development of methods regarding the reconstruction of precipitation maps. The first question looking to be answered is whether reconstruction is feasible when one is presented with a specific scenario of cellular RSL data. This issue is similar to the reconstruction of a random continuous time signal by a sampled time series. The Nyquist–Shannon sampling theorem states a criterion (that relates the bandwidth of the sampled signal and the sampling rate that yielded the time series) that when met, guarantees reconstruction without loss of information. The above question is a broad variation of that.

As an expansion of the above goal, one may ask how this research may guide system design. In that spirit, my work also concentrated on telling how a system may be designed so to allow it to successfully reconstruct a precipitation map.

One more research aspect in my work is the statistical modeling of microwave-links. While this is a consequential aspect that is simply meant to serve the main solution, it provides a novel study of the spatial distribution of microwave-links in various environments. Microwave-links are in fact random lines. They possess random lengths, random orientations, and are located randomly on the surface. In our context of rainfall monitoring, they present random projections (Sendik and Messer, 2012).

In this thesis I suggest mathematical criteria in the form of a diagram. These criteria dictates whether “good” reconstruction of the rain-field is expected. It incorporates within it the relevant parameters of the problem (e.g. how many links are available, what is the field’s spatial resolution, etc.), thus it is able to address every scenario individually. The term “good” reconstruction is also defined.

Moreover, I present a method for optimizing the reconstructed rain-field’s spatial resolution. This regards to the issue of system design, trading-off high-resolution and reconstructions error.

I should stress that I am concerned with the relationship between an instantaneous microwave measurements set and the corresponding rain-field. In other words, I am not studying
the time evolution of the rain-field. This work revolves around instantaneous reconstruction rather than prediction.

One should also note that this entire work revolves around the case where the power-law relating the rain-field and the attenuation field is the simple linear relationship presented in equation (1.3.3). As a reminder, this case relates to \( b=1 \), suitting wavelengths of about 1 [cm], i.e. a frequency of about 30 [GHz]. This case is appropriate to current cellular standards. The significance of this case is that one may regard the signal processing done in this work as being done either on the attenuation field or on the rain-field. The two differ by a scalar. Keep that in mind as throughout the report I interchange the terms “attenuation field” and “rain-field”. The probability of reconstructing a rain-field is identical to reconstructing the corresponding attenuation field, and the sparsity of the rain-field is identical to the sparsity of the corresponding attenuation field. On the other hand, if \( b \) were other than 1, then the entire work would be done on the attenuation field, as it is the attenuation field that the random microwave-links actually sample. The challenge would then be relating the reconstructed attenuation field to the corresponding rain-field.

1.7 Thesis’s Report Structure

1- Chapter 1 presents the background for rainfall monitoring, motivation, and the idea behind applying a backhaul network of microwave links to rainfall monitoring. It closes with the problem statement.

2- Chapter 2 discusses the technical theory that this thesis relies upon, Compressed-Sensing.

3- Chapter 3 formulates the mathematics behind the random microwave measurements and presents it in a specific linear form. That form corresponds to the technical theory from chapter 2. This paves the way for the solution to the main problem in the problem statement.

4- Chapter 4 addresses the statistical exploration of the microwave-links, the observations matrix, and rainfall sparsity. This allows for statistical modeling so to serve the solution approach from chapter 3.

5- Chapter 5 presents the results.

6- Chapter 6 discusses the results, their use, and concludes the work.
2 Surveying Compressed-Sensing

The prior work and research that this thesis capitalizes-on divides to two. The first one is the domain field and the other is the technical field. The domain field is rainfall monitoring via microwave-links. Chapter 1 discusses that domain. The survey in this chapter addresses the technical theory that was chosen to constructing the solution.

The technical field that I explored is called Compressed-Sensing. Being a very popular field for research in the last 10+ years, Compressed-Sensing deals with reconstructing signals that were sampled by very few and perhaps not uniform samples, like in our case. The survey I conducted gave me a clear enough image of the mathematical background and the appropriate applications it is used for, thus helping me understand how to apply it to my research. Four meaningful papers worth pointing out are (Donoho and Huo, 2001; Candes and Romberg, 2005; Donoho, 2006; Donoho and Tanner, 2009). Moreover, I took a class in the subject (Digital processing of single and multi-dimensional signals, 0510.6201, Prof. Alexander Bronstein, School of Electrical Engineering, Tel Aviv University, spring 2015).

2.1 Facing Partial Observations - Two Related Problems

2.1.1 Problem 1 - System Synthesis

Let us consider a compression-like need, where one may want to capture phenomena, e.g. time signals or images, and is interested in minimizing the amount of data necessary. The motivation may be to save memory, minimize communication bandwidth, or optimize device performance (i.e. achieve higher performance without increasing memory/bandwidth). A common example is a sinusoid. A sinusoid is fully captured by 3 variables, its amplitude, its frequency, and its phase. For instance, if one were to design a system where the typical signals are harmonics, the optimal compressing method may be to express the signals in the frequency domain. The key to explaining this compressed representation of harmonics is to understand that harmonics are sparse in the frequency domain, their energy is concentrated in few bins while the rest are “empty”, zeros. The problem is more intriguing when there isn’t a clear domain that naturally expresses the entire relevant signal space. The questions to ask are whether there is a domain in which the typical phenomena are sparse, how to find it, and how to go about the sampling method and the extraction. Compressed-Sensing theory provides insights to these three questions.

2.1.2 Problem 2 – Data Reconstruction from Partial Observations

The second problem to consider is when one attempts to retrace phenomena which aren’t fully observed. For example, Nyquist’s theorem defines sufficient conditions for optimal reconstruction of frequency band limited signals. The conditions demand a specific sampling scheme (uniform, point samples) with a critical sampling frequency. In many other real-world problems, one may find that the available observations are projections of the signal on a set of deterministic/random vectors (which aren’t deltas, thus not suiting Nyquist’s), and/or the original
signal isn’t band limited. The questions to ask are, how to best estimate the original signal, and what would be the estimation error. Compressed-Sensing theory provides insights these two questions.

2.2 The Element of Compressed-Sensing

The two problems have in common the need to optimally reconstruct the original signal from partial observations and to understand the reconstruction error. Ours is the second problem. We have no choice about the sampling scheme. The projections of the rain-field over the random link-lines are dictated to us. Using a-priori assumptions about the rain-fields and information about the sampling topology, a method can be designed to reconstruct the rain-field and to monitor the expected estimation error.

Let’s consider the case of observing signal \( x \) of length \( N \) through a linear operator that is modeled by matrix \( M, \) \( m \times N \), yielding the observation signal \( y \), as described by equation \( y = Mx \).

Immediately two different cases come to mind: 1) \( n \geq N \), the problem is over-determined and is well known, 2) \( n < N \) in which case classical linear algebra indicates that the linear system, \( y = Mx \), is underdetermined and has either infinitely many solutions, or no solutions (depending on the matrix \( M \)). So in case 2, lacking additional information, one can’t recover any unique \( x \) from \( y \). As previously mentioned, Nyquist’s contribution to such scenarios demands prior information about \( x \)’s band width. Thus, it seems only reasonable that there exist other conditions under which signal reconstruction is feasible even with the stingy sampling described above. They do exist and are what Compressed-Sensing revolves around. Moreover, efficient algorithms for such reconstruction do exist. As mentioned before, the underlying assumption which makes all this possible is sparsity.

Sparsity of a signal tells the count of its non-zero components (the gross amount). If \( x \) is \( k \)-sparse, then it has at most \( k \) nonzero coefficients. As it turns out, many practical signals are redundant in the sense that they are well approximated by sparse signals, when applying an appropriate “bridging” transformation. This explains why compression schemes such as JPEG, MPEG, or MP3 work so well in practice for images, videos, and audio, respectively. Each of these schemes relies on the sparsity of the signal in an appropriate domain, and achieves compression by simply storing the largest domain coefficients. This allows for the other coefficients to be set to zero, thus leaving much room for compression.

Note that a major difficulty stems from the uncertainty regarding the locations of the nonzero entries within the vector \( x \). If it was known which elements are the non-zeros elements, one would simply eliminate unnecessary columns from the matrix \( M \) and thus not sample entries of \( x \) which are known to be zeros.

Assume the information content of \( x \) is much smaller than its length (otherwise compression would not be relevant). This would hint that the required amount of measurements should be relative to this information content rather than the signal length.

Concluding the motivation for Compressed-Sensing, two questions belonging to the two problems introduced in this section come to mind:
• How should one design the linear measurement process? In other words, what matrices \( M \) are suitable?
• How can one reconstruct \( x \) from \( y = Mx \)? In other words, what are the efficient reconstruction algorithms?

These two questions are clearly related to each other as the reconstruction algorithm needs to take \( M \) into account. Note, however, that in our application there is little control over the design of \( M \) as it represents the links' line projections that are dictated by the problem's setup.

The availability of reasonably fast reconstruction algorithms is essential. This virtue has a big role in the attention brought to Compressed-Sensing. The first algorithmic approach coming to mind is \( l_0 \)-minimization. As \( \|x\|_0 \) marks the number of nonzero entries of a vector \( x \), it is natural to try to reconstruct \( x \) from the following optimization problem:

\[
(P_0) \quad x^* = \underset{x}{\text{argmin}} \|x\|_0, \quad \text{subjected to } y = Mx.
\]  

Which means, we could combinatorially search for the sparsest vector that suits the constraint \( y = Mx \). Meaning, out of all the vectors that satisfy this constraint, we pick the sparsest, thus fulfilling the \( l_0 \) minimization of \( x \). But, as suggested, this \( l_0 \)-minimization is NP-hard in general (Natarajan, 1995).

As an attempt to relax the problem and suggest an alternative to the minimization criterion, one may notice that the \( l_0 \) norm yields a function that isn’t convex. Thus, one may wonder what is the “closest” convex norm to \( l_0 \) that \( l_0 \) can be replaced with (convex meaning that the function \( f(x) = \|x\|_p \) is convex). Perhaps it will yield a solution that would also be a solution to the \( l_0 \) problem.

Fig 2.1 presents a few common norms. By observing these norms, one can get an understanding of the relationship between \( p \), and \( \|x\|_p \)'s shape. The \( p \) that is closest to 0 and
also allows convexity is \( p = 1 \).

![Figure 2.1 - Unit contours of several \( l_p \) norms. One can notice that for \( p \geq 1 \) the shape is convex.](image)

So, let's focus on \( l_1 \).

\[
(P_1) \quad x' = \arg\min_x \|x\|_1, \quad \text{subjected to } y = Mx. \tag{2.2.2}
\]

Basis pursuit is a known and even well-studied method for this \( l_1 \)-minimization. Having that the \( l_1 \)-norm corresponds to a convex function, \( P_1 \) is a problem that can be solved with efficient methods taken from convex optimization. In (Donoho and Huo, 2001; Elad and Bruckstein, 2002; Fuchs, 2002; Tropp, 2003; Tropp, 2004) the authors show criteria for when a \( P_1 \) problem has a unique solution that is also the desired unique solution to the \( P_0 \) problem.

Now let's look at how the observations matrix's design can be used to quantify the ability to reconstruct signal \( x \). As the challenge of designing a matrix that is optimal for Compressed-Sensing is great, breakthroughs have been achieved by studying random matrices (Candes and Tao, 2006; Mendelson, et al. 2008; Baraniuk et al., 2008). Some common examples are Gaussian matrices with entries that are independent random variables following a standard normal distribution, and Bernoulli matrices with entries that are independent random variables taking on the values +1 and −1 with equal probability. A key result in (Baraniuk et al., 2008) states that for
an \( n \times N \) Gaussian or Bernoulli matrix, \( \mathbf{M} \), it is with high probability that any \( k \)-sparse vectors \( \mathbf{x} \) can be reconstructed from \( \mathbf{y} = \mathbf{M} \mathbf{x} \) using a variety of algorithms while the following criterion is met:

\[
\frac{n}{k} \geq c \ln \left( \frac{N}{k} \right)
\]

(2.2.3)

where \( c > 0 \) is a constant.

### 2.3 Compressed-Sensing’s Contribution to this Thesis

Consider the need for a generic tool to be applicable to a greater set of measurement matrices and will provide the appropriate bound for reconstruction. Let’s observe \( \mathbf{x} \), a \( k \)-sparse vector of length \( N \). It is measured by \( n \) measurements, i.e. \( \mathbf{M} \) is \( n \times N \). Donoho and Tanner suggest in (Donoho and Tanner, 2009), that under certain conditions, there is a curve (diagram) that describes when the vector \( \mathbf{x} \) can be reconstructed.

Figure 2.2, taken from Donoho and Tanner’s work, shows the resulting diagram when applying the method to a Gaussian measurement scheme, meaning, \( \mathbf{M} \) is Gaussian and described in section 2.2. The diagram tells between the scenario that will allow reconstruction (under the curve) and that will not (above the curve). One of their greatest achievements was to show that the ability to reconstruct depends solely on two parameters: \( \delta = \frac{n}{N} \), and \( \rho = \frac{k}{n} \). One should note that another major achievement is an analytical proof dictating that this curve is the reconstructability bound for the Gaussian measurement scheme having \( N \rightarrow \infty \).

The diagram in figure 2.2 is actually one of several curves derived by Donoho and Tanner. It is the one that corresponds to \( \mathbf{x} \) taking on continuous value that may be both negative and positive. That is the only one that we are concerned with in this work. The reconstructed field takes on
both negative and positive values due to the sparsifying transform that applies to it (as will be seen in chapter 3).

The diagram shows that when the given $\rho$ and $\delta$ are such that the point $(\delta, \rho)$ is under the curve, then for a “large” $N$, it is with overwhelming probability that convex optimization will recover $x$ exactly. Moreover, when $(\delta, \rho)$ is over the curve then for a “large” $N$, it is with overwhelming probability that convex optimization will fail.

What about non-Gaussian matrices? In their research, Donoho and Tanner showed that a variety of random matrices exhibit the same behavior. Meaning, the same curve applies to them as well, separating the success and failure phases. Their entire contribution revolved around empirically validating various measurement ensembles. They thus presented a validation scheme that can be exploited to examine other measurement schemes. In the following chapter I will show how Compressed-Sensing is applied to our scenario, and how Donoho and Tanner’s method can be used to construct an appropriate success diagram.
3 Mathematical Formulations for Modeling and Reconstruction

This chapter deals with the mathematical modeling of the measurements process. It first describes the physical-mathematical formulation of a single link’s measurement. Then it presents an integrated model of all link measurements. Finally, it describes the significant part by deriving a mathematical notation that allows for applying results from the theory of Compressed-Sensing and provides the wished solution. Those farther methods and outcomes will be later presented in the results chapter.

3.1 Continuous Link Model

First, let’s take a look at the continuous model. It will not be the focus of the main analysis but this mathematical model will then be converted to a discrete model, allowing computations.

The measurement of a signal’s RSL along a path can be modeled as integration along a straight line stretching between the two base-stations, summing over the field of attenuation:

As formulated beforehand, the attenuation fields:

\[ X(r) \simeq aR(r)^\beta, \]

So the total attenuation that the signal endures along a link is,

\[ y_{[dB]} = \int_{\text{link}} Xds = \int_{\text{Entire Region}} X(r)\Lambda(r)dr_xdr_y \] (3.1.1)

where \(y_{[dB]}\) is the RSL value captured for that link, \(\Lambda(r)\) is a 2-D delta function that represents the link, and \(r\) is the position vector in the plane. Figure 3.1 depicts this phenomenon.

![Figure 3.1 - Portraying the process of capturing a RSL along a link. The left field is the attenuation field \(X(r)\) (which corresponds to the rain-field via the power-law), and it is multiplied by a delta-function representation of the link \(\Lambda(r)\). The final product is then summed over both dimensions.](image)
Conversely, the above integral can be written as an inner product:

\[ y_{[ab]} \equiv \iint_{\text{Entire Region}} X(r)\Lambda(r)dr_xdr_y \equiv \langle X(r),\lambda(r) \rangle \]  

This representation leads us to the discrete model.

\subsection{Discrete Link Model}

Our need is to recover the attenuation field with a certain practical spatial resolution. As much as the continuous model may be intriguing, our means are digital and we require a solution of finite spatial resolution.

Let us then observe our continuous measurements and relate them to a discrete field model. Let the continuous plane be partitioned to \( N \) pixels. Since the region of interest is rectangular, \( N \) is the product of the sums of pixels along the length and width of the region (figure 3.2).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.2.png}
\caption{Discretizing the continuous attenuation field. a – the continuous attenuation field, b – a grid with the wished resolution, c – the resulting discrete attenuation field.}
\end{figure}

Figure 3.2c portrays the matrix \( X \). This matrix is a discrete representation of the rain-field, or conversely, the attenuation field (the two being proportional to each other by equation (1.3.3)).

Our goal is to find the optimal approximation of the discrete field of resolution \( N \) that would yield the given set of RSL observations.

Let it be mentioned that there are a few different methods to discretize the field. For instance, one may set the pixel value as the mean of the field’s value over the pixel’s area, or as the value of the field in some point in the area of the pixel. The approach presented here isn’t concerned with that since it is not asking how to best present a sampled field, it is concerned with the space of all discrete fields of size \( N \) and is looking for the one that is best associated with the given set of RSL measurements. This association is explained below.

Let us now define a relationship between the discrete field and the RSL set. Since the discrete field is to optimally describe the continuous field, there is a clear scheme for discretizing a link.
line that is appropriate. In every pixel the link passes through, the pixel’s value should be the length of the link’s overlap with that pixel, see figure 3.3 for illustration. The resulting discrete representation of a link is matrix \( \Lambda_i \). It is depicted in figure 3.3b.

![Figure 3.3 - Discretizing the link. a – a single link of length 3.5 pixels, b – its discrete representation.](image)

Now let us again consider the inner product approach. Consider an operator that reshapes a matrix into a single column, \( CR\{\cdot\} \). This operator is portrayed here for a case of \( N=12 \):

\[
\begin{bmatrix}
X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\
X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\
X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\
\vdots & \vdots & \vdots & \vdots \\
X_{3,4} & & & \\
\end{bmatrix}
\]

Based on this operator, let \( \lambda_i \) and \( \mathbf{x} \) be defined as column representations:

\[
\lambda_i = CR(\Lambda_i),
\]

\[
\mathbf{x} = CR(\mathbf{X}).
\]

So the RSL measurement along the \( l \)–th link will then be:

\[
y_l = \lambda_i^T \mathbf{x}
\]

(3.2.1)
For convenience, let us normalize the link vector to sum up to 1. Since a link vector sums up to the link’s physical length, and since we know the length of each link, then let us define:

\[ \lambda_i^n = \frac{\lambda_i}{1_N \cdot \lambda_i} \]

and

\[ y_i^n = \frac{y_i}{1_N \cdot \lambda_i} \]

Here \( 1_N \) is a row vector of length \( N \) filled with 1’s. So the denominator is actually the length of the link. We now have:

\[ y_i^n = \lambda_i^{nT} x \]  \( (3.2.2) \)

Notice that this normalization does not change the result since the scaling is applied to both the link \( \lambda_i \) and the observation \( y_i \).

### 3.3 Full Discrete Integrated Model

Based on the above, the optimal field \( x' \) can be defined as the one that minimizes

\[ y_i^n - \lambda_i^{nT} x' \]

We are looking for one unique solution that will suit all available link observations simultaneously, so let’s examine the complete equation of all the links.

#### 3.3.1 The Complete Equation: \( y^n = A^T x \)

The task of finding the optimal discrete field can be formally described like so:

Given the measurements vector \( y^n \), find \( x' \) such that,

\[ y^n = A^T x' \]  \( (3.3.1) \)

Here \( A = [\lambda_1^n | \lambda_2^n | \cdots | \lambda_N^n] \), having a column for every normalized links vector.

This problem description may not be sufficient in the case that there are less links observations than unknowns (the length of \( x \)).

Section 4.2.2 surveys the structure and properties of matrix \( A \).

#### 3.3.2 Solution Approaches

When evaluating our ability to provide a solution to problem (3.3.1), the immediate question that comes to mind is whether the rank of the links matrix, \( A \), is greater or less then the number of unknown variables \( N \).
It is interesting to notice that the rank of the matrix depends on $N$. For small enough $N$, meaning, small with regards to the number of links $n$, there are very few pixels that the plane is partitioned to, thus making $A^T$ a “standing matrix”, having more rows than columns, i.e. having more observations than unknowns, and offering a range of algorithms to find the optimal field $x'$ (for instance, a least-squares approach). On the other hand, for a large enough $N$, the rank tends to $n$, as known from linear-algebra. Figure 3.4 illustrates this concept.

![Figure 3.4- Characterizing the matrix's rank as a function of $N$. The blue line is the rank of the links matrix, $A$. The black line is $N=\text{Rank}$ (i.e. the unit curve). To the left of the black line the problem is overdetermined ($A^T$ is a standing matrix). The gray area is where the problem is underdetermined, thus lacking observations. This is an empirical result. All along this experiment the number of links (i.e. $n$) is 1000. So the height $A^T$ is 1000, and its width (i.e. $N$) is being changed, corresponding to a changing field resolution. When $N<<n$ the rank coincides with the black line, thus allowing a wide range of solution approaches. But for $N>n$, the rank tends to 1000 (which is $n$'s value). To conclude, in the cases we are concerned with where $n<N$, the rank of the matrix is nearly $n$.

If we were to be satisfied with a resolution that allows for an overdetermined linear equation, we would be able to suggest a reconstruction criterion for $x$, like the minimum Mean Square Error, and easily derive the known solution
\[ y^n = A^T x' \rightarrow (AA^T)^{-1} Ay^n = x' \]  

(3.3.2)

The question to ask is whether we have to settle for a low enough resolution, just so we will only deal with a simple problem as in (3.3.2), or can we apply more sophisticated methods that will allow for a higher resolutions?
3.3.3 The Challenge – an Ill-Posed Problem

From this point on, this thesis focuses on the problem dealing with an underdetermined problem, where \( A^T \) is typically a “laying matrix” having more columns than rows \( N>n \).

The underdetermined problem is definitely the more intriguing one, where for an observed area, there tends to be less links than desired pixels. There are two motivations for dealing with such cases. One, there is a need for higher spatial resolutions for the obvious reasons of being able to appeal to different applications that require more precise monitoring of rainfall. Two, science has no such data. We lack the ability to know what is happening in very high resolutions and the higher the resolution is, the more it’ll contribute to research.

We are faced with a problem in the form of a linear system, having more unknowns than equations. Classic linear algebra will not suggest any one particular solution and so we are forced to supply farther a-priori information about the field \( x \) and use it to constrain the infinite space of solutions in hope to find one unique solution.

3.4 Adapting Our Model to Suit Compressed-Sensing Theory

Compressed-Sensing offers an intriguing mathematical approach. If a transformation were to be presented that would transform the original attenuation field to some sparse field, and could be implemented as a matrix multiplication which is invertible, then we could utilize Compressed-Sensing to define reconstruction criteria for telling when and how the field could be reconstructed.

Let \( S \) be an invertible \( NxN \) matrix,

\[
y^n = A^n x = A^n S^{-1} Sx = M x_s
\]

\[
y^n = M x_s.
\]

(3.4.1)

\[
M = A^n S^{-1}
\]

(3.4.2)

\[x_s = Sx.
\]

So motivation is clear, find \( S \) such that would yield a sparse vector \( x_s \) while letting \( M \) qualify for certain reconstruction criteria in Compressed-Sensing. These criteria would then translate to criteria for the ability to reconstruct attenuation field \( x \) from the random links ensemble \( A \). In the following section I show what other desirable property \( S \) should possess.

The research done in (Donoho and Tanner, 2009) allows for one to explore new measurements ensembles and yield their ability to allow reconstruction when given \( N, n, \) and \( k \), being the number of unknowns, the number of measurements (equations), and the sum of non-zeros in \( x_s \), respectively.
3.5 Choosing a Sparsifying Transform

Donoho and Tanner’s results apply to the ability to reconstruct the sparse vector $x_s$. It is highly desirable to be able to relate these results to the ability to reconstruct the original vector $x = S^{-1}x_s$. That ability obviously depends on the matrix $S$.

The metric that measures reconstruction is the relative reconstruction error. We can define:

$$E_{\text{sparse, rel}} = \frac{\|x'_s - x_s\|_2^2}{\|x_s\|_2^2},$$

$$E_{\text{non-sparse, rel}} = \frac{\|x' - x\|_2^2}{\|x\|_2^2},$$

where the prime character, $'$, states the reconstructed vector.

A sufficient condition that will allow for the adaptation of Donoho and Tanner’s work is:

$$E_{\text{non-sparse, rel}} = E_{\text{sparse, rel}}. \tag{3.5.1}$$

This way, if both errors are always equal, then they share the same phase-transition curve. Replacing to error terms with their definitions gets:

$$\frac{\|x' - x\|_2^2}{\|x\|_2^2} = \frac{\|Sx' - Sx\|_2^2}{\|Sx\|_2^2} \Rightarrow$$

$$\frac{\|x' - x\|_2^2}{\|x\|_2^2} = \frac{\|S(x' - x)\|_2^2}{\|Sx\|_2^2}.$$ 

Defining $e = x' - x$,

$$\frac{\|e\|_2^2}{\|x\|_2^2} = \frac{\|Se\|_2^2}{\|Sx\|_2^2}.$$ 

If $x$ and $e$ can be any two vectors in $\mathbb{R}^{N \times 1}$, then we derive that there is a constant $\alpha > 0$ such that:

$$\frac{\|Sx\|_2^2}{\|x\|_2^2} = \frac{\|Se\|_2^2}{\|e\|_2^2} = \alpha^2.$$ 

Meaning, for every $x \in \mathbb{R}^{N \times 1}$,

$$\frac{\|Sx\|_2}{\|x\|_2} = \alpha \frac{\|e\|_2}{\|e\|_2}.$$ 

So, $S$ must be an isometry up to a level of scaling. Conversely, $S$ must be unitary up to a level of scaling. This is both a sufficient and necessary condition to fulfill (3.5.1).
3.5.1 Applying Haar Wavelets to Allow Sparsity

Image Processing and Signal Processing theories present a great deal of study on sparse images and signals. In (DeVore, 1992) we see that the Haar transform of an image is shown to provide an almost natural representation of natural images. Meaning, when transformed to the Haar domain, most of the image's information is preserved via very few coefficients. As will be demonstrated below The 2-D image Haar transform holds many favorable properties that make it a leading candidate for $S$ (Mallat, 2008). The Haar transform is easy to implement numerically, it is also linear, invertible, and preserves an image's norm (i.e. unitary), a property that was proved in the previous section to be sufficient for our solution.

Let $S$ be the matrix that preforms the Haar transform on the column representation of the original field $X$. Meaning, $Sx$ is the column representation of $\text{Haar}(X)$, $\text{Haar}(X)$ being the 2-D matrix transformation of the field $X$. $S$'s structure is detailed in Appendix A along with a proof that $S$ is unitary. Figure 3.6 presents a few examples of transforming tentative rain-attenuation images to the Haar domain.

Here I explain about the structure of the Haar transform. Haar transform is originally structured for a 1-D signal. The Haar transform of a 1-D signal is comprised of 2 parts, the approximation coefficients (i.e. coarse), and the detail coefficients (i.e. fine). This transform is easily adapted to 2-D signals. The Haar transform of a 2-D signal is comprised of 4 parts: approximation-approximation (AA), approximation –detail (AD), detail-approximation (DA), and detail-detail (DD). It is accustomed to present the Haar transform of an image as one image incorporating the four parts in some fashion.

Each of these four parts stands on its own, so if they are to be appended together to be presented in one image, there may be a few different ways to do so. I employed Matlab's (Natick, MA, USA) arrangement (portrayed in figure 3.5). Let $b[i,j]$ represent the Haar transformed image in coordinates $(i,j)$, then:

$$\begin{align*}
\text{AA} &= b(1:2:end, 1:2:end) \\
\text{AD} &= b(2:2:end, 1:2:end) \\
\text{DA} &= b(1:2:end, 2:2:end) \\
\text{DD} &= b(2:2:end, 2:2:end).
\end{align*}$$

It should be noted that the chosen arrangement does not change the results of the field reconstruction.
Figure 3.5 - The layout of the 4 Haar coefficient types. a – the original Lena image, b – Matlab’s Haar layout, c – another Haar layout which is also common.
Figure 3.6 - Portraying the sparsifying effect of the Haar transform on tentative attenuation-fields. Every row has a tentative attenuation-field on the left and its Haar transform on the right.

3.5.1.1 The Reconstruction Stages

By solving the $P_1$ problem:

$$ (P_1) \quad x_s' = \arg\min_{x_s} \|x_s\|_1 \quad \text{subjected to} \quad y^n = M x_s. $$

we are left with $x_s'$ which, by definition of the $P_1$ problem, is optimal from a $l_1$ perspective. We know that $x_s' = Sx$, so finally we write,

$$ x' = S^{-1}x_s'. $$

Donoho and Tanner’s phase transition tells us about the ability to reconstruct $x_s$. Since the Haar transform is unitary, we know that the same phase transition applies to the reconstruction of $x$.

The literature has a lot of research regarding the solution to the $P_1$ problem, a couple of major papers in the matter are (Donoho and Elad, 2006; Donoho and Tsaig, 2006).

Remark, appendix C presents experimentation with a transformation that isn’t fulfilling the unitary condition I derived. That experiment took place before I derived that mathematical condition, moreover, it was this experiment that lead me to seek mathematical conditions to begin with.
3.6 Concluding the connection between Compressed-Sensing and Microwave RSL Observations

It is thus formulated that our RSL observations can be presented as (equations (3.4.1)):

\[ y^n = Mx_s, \]

where \( x_s = Sx \) is a sparse representation of \( x \), \( M = A^T S^{-1} \), and \( S \) is a the Haar transform. Donoho and Tanner’s work can be applied to this equation to find the expected ability to reconstruct \( x_s \). Then, using \( x = S^{-1} x_s \), equation (3.5.1) tells us that the ability to reconstruct \( x_s \) is identical to the ability to reconstruct \( x \).
4 Microwave-Links and Rain-Fields: Statistical Studying and Modeling

As a preliminary step to calculations and simulations, it is necessary to study the statistics of microwave-links. This study is meant to capture the links’ random characteristics. Some clear geometrical and topological attributes are presented. These attributes are later used to generate synthetic links for the sake of simulating scenarios of rain-field measurements.

The spread of microwave-links is designed based on several considerations. When considering how to deploy base-stations, one can divide the factors to two, micro and macro. Micro factors would be those that make a minor difference on the location of the base-station. For instance, after deciding to position a base-station on a specific street block, one would consider micro factors dictating positioning the base-station on a specific building’s roof-top rather than another building’s. The macro factors, on the other hand, will dictate to the cellular providers what would be the amount of microwave-links to deploy in a large area (in the tens and hundreds of square kilometers), how to spatially distribute them, and how their lengths should vary. Insights regarding the design of backhaul networks can be found in (ECC Report 82, 2006). The claim that there is a random factor in the spread of microwave-links relies on both the micro and macro factors. The macro factors are addressed in this thesis since they affect the distribution of links. Spatial distribution of links can be divided to subsets of scenarios based on population density and topography.

![Figure 4.1- The distribution of microwave-links in France for three cellular providers: Bytel, Orange, and SFR (Christian Chwala, private communication).](image-url)
Figure 4.1 shows the distribution of microwave-links in France for three cellular providers: Bytel, Orange, and SFR. By taking a look at the map one can get a feel for the volume and distribution of links. It is clear that the distribution is in the form of dense clusters. The major cluster (latitude=49, longitude=2.5) is of the Paris area. When studying France some more one would find that the rest of the dense clusters are too of cities, i.e. concentrations of population. So, this links map gives us an intuition for the macro factors regarding distribution and volume. The volume is higher in heavier populated area.

4.1 Spatial Distribution

Literature and studies led me to discover that the main factor for deciding on the amount of microwave-links for a given area is the density of population and the topography. The common categories that tell between spatial distributions are urban (most dense), suburban, and rural (least dense) (Zhang et al., 2013; Yifan, et al., 2015). In (Amaldi et al., 2003) one may find the considerations of the environment type from a base-station capacity perspective. Table 4.1 shows results of base-stations density studies, next it will be explain how they relate to links studies.

Table 4.1 Base-station densities based on environment type. One can notice how base-stations densities depend on population densities. Since links topologies are such that the number of links is nearly identical to the number of base-stations, this table also reflects the spatial densities of links (Yifan, et al., 2015).

<table>
<thead>
<tr>
<th>Region</th>
<th>Area [Km²]</th>
<th>BS Amount</th>
<th>BS (~ Links) Density [1/Km²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Most Dense City (A)</td>
<td>60×40</td>
<td>6,251</td>
<td>2.604</td>
</tr>
<tr>
<td>Second Densest City (C)</td>
<td>30×50</td>
<td>1,911</td>
<td>1.274</td>
</tr>
<tr>
<td>Third Densest City (B)</td>
<td>40×40</td>
<td>977</td>
<td>0.611</td>
</tr>
<tr>
<td>Rural</td>
<td>200×200</td>
<td>12,691</td>
<td>0.317</td>
</tr>
</tbody>
</table>

In order to apply the study of base-stations to our study of microwave-links we need to find the connection between the number of base-stations and microwave-links. There are four topologies for connecting base-stations and forming microwave-links. Figure 4.2 describes the four. If we were to look into each one of the four topologies we would find that the number of links is nearly identical to the number of base-stations. Thus, the densities derived in the base-station studies in table 4.1 apply to microwave-links just as much.
This distinction between distributions of microwave-links based on environment types is significant to the method derived in this study. This thesis makes an assumption that microwave-links are being distributed uniformly, i.e. not “favoring” a specific piece of the spatial region over the rest. More specifically, if we were to define each link’s position based on the position of its center, then we’d say that the centers are distributed uniformly. Thus, it was important to get an understanding for when one may assume such uniform distribution. The assumption made in this work is that for a region of homogeneous nature (e.g. just urban, or just rural), the spatial distribution is of a uniform nature. This assumption is based on and supported by the studies cited in the last paragraph, particularly (Zhang et al., 2013). Thus, every non-homogeneous region could be partitioned to several homogeneous regions, and the calculation will be done separately.

4.1.1 Partitioning a Region to Sub-Regions of Uniform Links Density

Based on the above, one sees that when approaching a region of diverse environments such as rural, suburban, and urban, it is recommended to partition it to homogeneous sub-regions. The results of this study, as will be shown, are concerned with the number of links in the region of interest. It is assumed that those links’ centers are spread somewhat evenly, meaning, not clustered together. If the region was not partitioned as intended and so the links were clustered together, then their spatial distribution would need to be addressed more specifically in order to evaluate reconstruction potential.

Another guideline for partitioning a region is the nominal area that will be appropriate for capturing any relevant phenomena. Typical rain-clouds over Israel tend to reach an area of up to 10x10 [Km²] (Karklinsky and Morin, 2006). It is recommended to maintain a minimum of such area. In this work I examined regions that are 10x10 [Km²].

4.2 Length and Orientation

Besides the locations and volume of microwave-links, the characteristics of the link also affect the measurement. These characteristics are the orientation and length of the link.

The study made here on links data given for the state of Israel shows that in any type of region studied, the orientation of links takes on any angle with equal probability. Meaning, the direction
of microwave-links is distributed uniformly without relation to the type of population density. Figure 4.3 portrays this conclusion. Three regions were isolated for analysis. Turn to appendix D for more details on these regions.

![Histograms of Link Angles](image1)

**Figure 4.3** The distribution of link angles. The angle is uniform and doesn’t depend on the type of environment: Top-left: All Israel, top-right: Ramat-Hagolan as rural, bottom-left: Hasharon as suburban, bottom-right: Tel-Aviv as urban.

Note that there is prior work that addressed the distribution of links orientations and lengths. In his master thesis, Sendik (Sendik, 2013) divided Israel to four parts based on latitude (Israel stretches from latitude 29.502888 to 33.334972) and did not pick regions based directly on population density.

The length of microwave-link, unlike orientation, is distributed non-uniform, and depends on the type of environment.

The distribution of lengths of microwave-links is non-uniform, it is exponential. This claim is portrayed in Fig 4.4.
The means in the above analysis correspond directly to the fitted exponential distribution. In exponential distribution, e.g. \( T \sim \text{Exp}(1/\theta) \), the probability density function is:

\[
f_T(t|\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{t}{\theta}} & t \geq 0, \\ 0 & t < 0, \end{cases}
\]

where \( \theta = \text{E}[T] \).

This explains the direct connection between the found means and the exponential fitting.

Table 4.2 concludes the empirical results for microwave-links in Israel.

<table>
<thead>
<tr>
<th>Region</th>
<th>Area [Km²]</th>
<th>Links Amount</th>
<th>Links Density [1/Km²]</th>
<th>Links Mean Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>All of Israel</td>
<td>22,770</td>
<td>3,624</td>
<td>0.16</td>
<td>3.54</td>
</tr>
<tr>
<td>Tel-Aviv (urban)</td>
<td>85.18</td>
<td>264</td>
<td>3.1</td>
<td>1.48</td>
</tr>
<tr>
<td>Hasharon (suburban)</td>
<td>235.54</td>
<td>124</td>
<td>0.53</td>
<td>2.5</td>
</tr>
<tr>
<td>Ramat-Hagolan (rural)</td>
<td>1341.15</td>
<td>98</td>
<td>0.07</td>
<td>5.51</td>
</tr>
</tbody>
</table>
4.2.1 A Proxy for Environment Density: Links’ Mean Length

An interesting connection is noticed between the density in some environment and the lengths of the links. In all environments, the links’ lengths are distributed exponentially and their mean length is representative of the density. The denser the environment (urban), the smaller the mean length is. Thus, when characterizing an environment later in this thesis, it will be done by choosing a proper links density and a proper mean length.

Let it be mentioned that the links’ density is of course also a clear proxy for the environment type.

4.2.2 The Links Matrix

Here I survey the characteristics of the random links matrix, $A$, defined in chapter 3. $A$’s characteristics are consequences of the microwave-links, and so $A$’s study suits this section. Note however, that the study of $A$ doesn’t serve a direct purpose in this thesis, unlike the studies of links and rain-fields. The study of links statistics serves a clear purpose of modeling sets of links, used to calculate and validate the solution to the reconstruction problem. The study of rain-field sparsity that will be presented later serves the same purpose, moreover, it validates one of the main assumptions in this thesis, that rain-field have sparse representation. On the other hand, $A$’s study isn’t used of constructing $A$. It is the synthesizing of microwave-links that construct $A$. The point of this section is to shed light on the structure of $A$, allowing the reader to perhaps get a better intuition.

Given the linear equation $y^n = A^T x'$ to be solved for $x'$, $A$ is the component relevant to study for suggesting a solution. If, for instance, $A$ is a full rank square matrix, then calculating its inverse yields a unique optimal solution:

$$A^{-1} y^n = x'.$$

As an effort to give some initial intuition on the components of $A$, let’s see how it is affected by the resolution parameter, $N$.

$A$’s elements take on values in the interval $[0,1]$. This is a result of the links normalization. Notice that in a continuous model, the area that a link-line covers can be defined as zero, as it is a line in $\mathbb{R}^2$. This explains that the higher the discrete resolution $N$ is, the more sparse the vector $\lambda^T_i$ is. This automatically means that the higher the discrete resolution $N$ is, the more sparse matrix $A$ is, being filled with many zeros and few positive entries.

Figure 4.5 illustrates the relationship between $N$ and the ratio between zero to non-zero components:
4.2.2.1 Statistical Dependencies between columns

As \( A = [\lambda_1^1 \mid \lambda_2^1 \mid \cdots \mid \lambda_n^1] \), every column is comprised of one link. Thus, an element in a certain column is statistically independent of any other element in any other column. Also note that the order of the columns is arbitrary. The columns can take on any order without changing the problem or the solution, given of course that the y’s RSL measurements are shifted accordingly.

4.2.2.2 Statistical Dependencies between rows

As all elements in the same column correspond to the same link, an element in a certain column statistically depends on other elements in the same column. It is clear that if \( x_a \) and \( x_b \) are neighboring pixels in the physical plane, and if we know that link \( l \) passes through \( x_a \), it implies that this link also passes through \( x_b \) with a higher probability than if it didn’t pass through \( x_a \) (see figure 4.5):

\[
P(x_a \text{ belongs to link } l \mid x_b \text{ belongs to link } l) > P(x_a \text{ belongs to link } l).
\]

Meaning,

\[
P(x_a > 0 \mid x_b > 0) > P(x_a > 0).
\]

This inequality dictates statistical dependence.

Moreover, the farther the pixels are apart in the physical plane, the less probable it is that they both belong to the same link. These claims are simply based on the fact that links are types of clustered pixels in the plane.
Thus, the matrix’s components are statistically dependent. Since the components in the matrix are in different arrangements, then the statistical dependency is no longer just a matter of who is whose neighbor (as seen in figure 4.6). But this section is not concerned with precisely describing these dependencies. The point of this section is to simply illustrate that the entries along a certain column of $A$ are in fact statistically dependent.

### 4.2.2.3 Statistical Distribution of the Elements of $A$

**Elements are Identically Distributed**

Since we assume that the region of interest is chosen such that it is homogeneous and represents one environment type (e.g. urban), we also assume that the link’s centers are distributed uniformly. In different words, every point in the region is equally probably to encounter a link. Translating this to elements of $A$, each element is equally probable to belong to a link. Thus, all the components of $A$ are identically distributed.

**The Distribution of the Elements**

Naturally, two major factors affect the distribution law of the elements, the lengths of the links and the resolution $N$ (i.e. the partition of the physical region). The size of the physical region would of course play a role but in this thesis I focus on an area of 10X10 [Km$^2$]. Figure 4.7 presents how the distribution of the elements varies as the resolution and the mean length change.
Figure 4.7 – Histograms of elements of matrix $A$. The bar most to the left for the frequency of “0” elements is clipped as its height is close to 1. One can notice how the partition of the region to more pixels and the length of the links affect the values’ frequencies.

4.3 Rain-Field Sparsity Study

When looking to evaluate the ability to perform rain-field mapping over a defined relevant area using the method discussed in this thesis, one should evaluate the sparsity of the rain-fields typically over that region. One way that can be done is to study weather-radar images and evaluate the fields’ sparsity.

4.3.1 Applying Weather Radar Images to Evaluate Rain-Field Sparsity

The Swedish Meteorological and Hydrological Institute (SMHI) published rain-field radar-images of 288 instances (http://opendata-download-radar.smhi.se/api/version/latest/area/sweden/product/comp/2016/8/13.zip?format=png). While every image is of all of Sweden, only a portion of each image was taken so to capture a partial region (figure 4.8). This region is captured only for the sake of studying the rain-field. If we were to define a region for microwave-links reconstruction then we would not choose a region with sea, but just over land.
SMHI’s image resolution is such that in the chosen region there are 256x256 pixels. The region is 512x512 Km², thus making the resolution $4 \text{ [Km}^2/\text{pixel]}$. The 288 radar images were captured, transformed by Haar’s, and checked for sparsity. Let it be noted that they were not scaled to transform from rain values to attenuation values as the sparsity evaluation is invariant to scaling. The histogram of the sparsities is in figure 4.9.

Figure 4.8 - The region of study. 288 images were studied for sparsity.

Figure 4.9 - A histogram of sparsities for typical rain-fields over south Sweden.
As seen, the transformed fields are indeed sparse. All of them demonstrate sparsity of 33% or better. This result supports the motivation to turn to Compressed-Sensing and attempt sparsity-based methods.

4.4 Study Conclusions

In this chapter are empirical statistical results of microwave-links and rain-field sparsity. The links results characterize the links’ spatial-distribution, lengths, and orientations. They are based on links in Israel for a single cellular provider, Cellcom.

Three attributes are fitted with statistical models. The spatial-distribution is modeled as a uniform distribution with a variable density dependent on the environment type. The lengths are modeled as exponential random variables, with a variable mean length also dependent on the environment type. The orientation angles are modeled as random variable distributed uniformly across all possible angles \([0, \pi]\).

The sampling matrix, \(A\), is studied. Its dimensions and its elements are being analyzed.

Rain-field sparsity is being studied empirically using a set of 288 radar images taken consecutively by weather radars over south Sweden, provided by SMHI. The study finds for the studied case that the typical rain-field is indeed sparse in the Haar domain.
5 Calculations and Simulations

Following the two previous chapters which presented a mathematical modeling for the microwave observations (chapter 3) and a statistical modeling for the nature of these observations, i.e. the microwave-links (chapter 4), this chapter integrates them to form the suggested solution for this thesis’s main question, deriving a reconstruction feasibility criteria. First, simulations are constructed so to generate synthetic data for the calculations. Links are simulated with correspondence to the study of microwave-links. Following that, matrix $A$ is constructed, and eventually matrix $M$ is calculated based on (3.4.2). Second, the attenuation-fields are simulated with correspondence to the assumption of sparsity. Finally, the main simulation is described. It is the method that Donoho and Tanner developed in their work (Donoho and Tanner, 2009). It is this method that will generate the solution to the question whether a certain scenario allows for reconstruction.

5.1 Simulating Links

The main part of this thesis was done using simulated links. The motivation behind using simulated link rather than actual given links revolves around:

1- Having a controlled study: When using simulated links we can account for every factor in their nature.
2- Allowing reliability of the results: The results rely completely on the controlled assumptions, and are less-prone to overlooked factors that may exist with real data.
3- Robustness: When simulating links we are not limited to an available set. We may generate as many links as necessary for statistical thoroughness.
4- Sensitivity analysis: We are able to tweak links parameters and evaluate their effect.

Links are to be simulated with two key factors. The first one is the links’ mean-length, it is a physical parameter. The second is the spatial resolution, it isn’t physical, it is a computation parameter. It is the partition of the region to pixels. Since a link is represented as an array of many zeros and few non-zeros (where the link lays), then the spatial resolution dictates how many elements will be in that array, i.e., how fine the links array will be.

One may wonder why spatial density of links isn’t a factor. All sets are generated so to account for all relevant spatial densities. All sets are generated with a “very high” number of links (as explained farther down), and when different spatial densities are to be evaluated, a small part of the links set is actively being used. For instance, for an area of 100 [Km$^2$], a set may be generated with 10,000 links. When simulating a reconstruction iteration, perhaps only 2000 links will be randomly selected, and the fact that there is a “very high” number of links will allow for Monte-Carlo repetitions of the scenario with different 2000 random links. Here “very high” refers to a number so high that will allow the maximal number of links to be randomly selected out of the set. The complete opposite would be to limit the set to have “just enough” links, thus allowing only one manner for selecting the maximal number of links out of that set (i.e. select the entire set). The alternative to this random choice of links from a pre-calculated set would be to randomly regenerate the relevant number of links every time on every iteration. The only
downside to the latter is that it is more computationally intensive, as links are constantly being generated during the reconstructions simulations.

5.1.1 The Creation of a Synthetic Microwave-Links Set

Pre-set variables (variables that are constant throughout the run):

1. Region Area:
   Chosen the same for all simulations in this thesis, square, 10x10 [Km²].

2. Spatial Resolution - \(N\):
   This parameter varies between links sets. As low as 32x32 pixels (i.e. 0.3125 \(\times\) 0.3125 [Km\(^2\)/pixel]), and as high as 256x256 pixels (i.e. 0.0391 \(\times\) 0.0391 [Km\(^2\)/pixel]). Different set are generated for different resolutions. The motivation is to focus on the highest \(N\) possible so to correspond with Donoho and Tanner’s results.

3. Environment Type – Mean Links Length:
   This parameter varies between links sets. Simulating a different environment means choosing a different mean length. Values typically range from 1.5 to 10 [Km²].

4. The number of links to generate \(n_{set}\) (should be greater than \(N\)):
   For instance, assume \(N\) is 32x32=1024 pixels. When this entire set will be used in the simulation, the scenario with the largest amount of links will be when \(\delta = 1\), which means 1024 links are being chosen to sample the field at that scenario. This scenario will repeat itself many times (for Monte Carlo aggregation). Thus we would like the set of links to have enough links to allow random choices of 1024 links out of it. If, say, we’ll generate \(n_{set}=1100\) links, we’ll have \(\frac{1100}{1024}\) \(\approx\) \(5 \cdot 10^{118}\) ways to choose 1024 links out of that set.
   In fact, to allow great variability, the chosen set size is always \(n_{set}=1.5N\).
   Note that the links set has the most links necessary to reflect any links density. Thus, by generating enough links to correspond to urban density, we of course also have enough to correspond to rural.

The microwave-links set generation:

1- Generate an array of \(n\) objects, each with a length value:
   Each length is drawn randomly from an exponential distribution with the above pre-set mean length.

2- Assign each object an orientation:
   Each of the generated objects is assigned an angle drawn uniformly from \([0, \pi]\).

3- Assign each object a position:
   Each object’s beginning point is drawn uniformly in a continuous square of size \(\sqrt{N} \times \sqrt{N}\).
   Then the end point is defined by drawing a line based on the length and orientation of the object.
   Note that at this step, each link is defined by “continuous” measures, meaning, nothing has been rounded or quantized yet.
Calculate quantized pixels values:
In order to suit the discrete model, the link is being represented as pixels. Divide the $\sqrt{N} \times \sqrt{N}$ area to $\sqrt{N} \times \sqrt{N}$ squares, each will represent a pixel. Set 0 to a pixel that doesn’t have the link pass through it. For a pixel that the link does pass through, assign a positive value equal to the physical length of the link overlapping with the pixel’s region (i.e., overlapping with the square). See figure 3.3 for a graphical description.

Figure 5.1 Portrays the controlled variables.

![Figure 5.1 - Portraying the variables of the synthetic links. Left: spatial distribution, middle: lengths distribution, right: angles distributions.](image)

5.2 Simulating Rain-Fields with Variable Sparse Representation

The theory that Donoho and Tanner developed (Donoho and Tanner, 2009), when applied here, revolves around the sparsity of the rain-field. The theory assumes that in order to calculate the feasibility to reconstruct the rain-field, the only rain-field characteristic that is relevant is its sparsity in the transform domain. This reminds us of the famous Shannon-Nyquist theorem which says that the only signal characteristic that is relevant for determining whether perfect reconstruction is feasible is its band-width. Of course both methods require farther information about the measurement/sampling scheme, but these don’t regard the signal.

However, in their research, Donoho and Tanner do determine a certain distribution for the field. They assume the field, after being transformed to the domain where it is sparse, has values distributed uniformly in [-1,1] (Donoho and Tanner, 2009). Note two things: 1- this assumption applies only to the non-zeros values, 2- this assumption only regards to the values of the non-zeros, not their locations, as they are assumed to be positioned randomly in the region (i.e. a uniform spatial distribution).

One may wonder how different distributions of the non-zero values would affect the results, meaning, when the non-zeros are not distributed uniformly in [-1.1]. Recall Donoho and Tanner’s main results that this report revolved around, the phase-transition diagram as presented in figure 2.2. I’ve recalculated the phase-transition diagram for six different distributions. The first one being uniform in [-1,1], and five other. All six results were nearly identical, supporting an assumption that the theory is invariant to the values of the non-zeros, and strengthening the
The assumption that it is the sparsity of the field that is its main feature for reconstruction. The six results are plotted in figure 5.2.

![Figure 5.2 - Exploring the empirical phase-transition diagram for various distributions of the non-zeros in the sampled signal x. N=128x128 in all. Each figure's title describes the examined distribution. The difference between the six is negligible.](image)

### 5.2.1 Motivation to Simulating Rain-Fields

Following Donoho and Tanner's research, we wish to have a rain-field for each of the possible sparsity values, meaning, to have real rain-field images that correspond to every sparsity level. In practice, that isn't the case. There are many levels of particular sparsities that we don't have corresponding fields for. Thus, I chose to simulate fields, each field with a prescribed sparsity.

#### 5.2.2 The Creation of a Simulated Sparse Rain-Field

**Pre-set variables:**

1. **Region area:**
   - Chosen the same for all simulations, square, 10x10 [Km²].

2. **Spatial resolution - N:**
   - This will determine which resolution will be examined with this specific rain-field.
   - This parameter varies between simulations.
   - See the pre-set variables in section 5.1.1 for some more details about the values of N.
   - For any value the N takes, there is a corresponding links set that was pre-generated.

3. **Sparsity level:**
   - This will determine which sparsity will be examined with this specific rain-field.
Values range from 0.001 to 1. A sparsity of 0.001 means that 1 of 1000 pixels is non-zero.

4- **Sparsifying transform S**: As a sparsity level is set, it should correspond to some sparsitying transformation. $S$ is a matrix that performs the Haar transform as depicted in chapter 3.

The generation:

1- **Create the sparse field $x_s$**: Based on the wished sparsity level and the field size $N$, calculate $k$ which is the number of non-zero pixels:
$$k = \text{round}(N \cdot \text{sparsity level}).$$
Then, choose $k$ pixels arbitrarily (i.e. their position is distributed uniformly on the $\sqrt{N} \times \sqrt{N}$ grid). Assign Each one of these $k$ pixels values drown uniformly from the interval $[-1,1]$.

2- **Retrieve the rain-field $x$**: Calculate $x = S^{-1}x_s$.

5.3 The Main Simulation

5.3.1 Computational Resources for the Main Simulation

The calculations performed throughout this research were extensive. Over the last three years, up to three PCs simultaneously ran computations. As this research was coming together the computation tasks were more focused and efficient. In the last two years there was at least one PC performing computations. Some of the major simulations took several months each. It is estimated that the accumulated running time of all computers is about two years.

The PCs varied, the most powerful of which had a Windows 7 operation system, 16 cores, 64 GByte of RAM, and a 2.6 GHz processor. All processing was in Matlab (Natick, MA, USA).

The computations were optimized so to only compute the regions of interest. The critical calculations of this thesis are concerned with the area around the phase transition curve that tells between good and poor reconstruction, making the regions that are far from the bound quite predictable and of no interest. These regions are the lower-right and the upper-left corners of the $(\delta, \rho)$ plane.

5.3.2 Describing the Main Simulation

The product of Donoho and Tanner’s research is a simulation scheme that provides a reconstruction-feasibility diagram. The method is empirical. As they note in their work, besides a few “friendly” measurement schemes such as Gaussian and orthonormal matrices, the analysis of most schemes are too complex to solve analytically. The simulation’s goal is to fully scan all possible scenarios and to yield a diagram that tells when one may reconstruct the rain-field and when one may not. The significance of this diagram was explained at the end of section 2.3.
The simulation iterates over three variables: $\delta$, $\rho$, and the Monte-Carlo iterations.

**Pre-set variables:**

1. **Region area:**
   Chosen the same for all simulations, square, 10x10 [Km$^2$].
2. **Spatial resolution - $N$:**
   This will determine which resolution will be examined.
   Important remark, as Donoho and Tanner’s phase transition curve is proven analytically for $N\rightarrow\infty$, in order to validate that this thesis corresponds with their theory, the largest $N$ that still allowed computation was chosen (more in chapter 6). See the pre-set variables in section 5.1.1 for some more details about the values of $N$.
3. **Environment type:**
   This affects the links set being used. For instance, if a dense urban environment is chosen, then the links will have mean length of 1.5 [Km]. As explained in section 5.1.1, the environment’s effect on links density is being addressed when picking the number of links to be used in every simulated scenario. This affects the links density as the region’s area is fixed and does not vary.
4. **Number of Monte-Carlo iterations:**
   This varies between simulations. Can be as low as 8, and as high as 256.
   This value was experimented with in prior simulations and was chosen to allow for the results to converge. Note that this the number of necessary Monte-Carlo iterations depends on the field size, $N$. For smaller $N$’s, more Monte-Carlo iterations are necessary to allow convergence.
5. **Number of partitions for $\delta$, ranging in [0.1,1].**
6. **Number of partitions for $\rho$, ranging in [0.01,1].**
7. **Sparsifying transform $\mathbf{S}$:**
   Deciding which sparsifying transform will be applied.
   $\mathbf{S}$ is a matrix that performs the Haar transform as depicted in chapter 3.

The simulation:

For each Monte-Carlo iteration do:
For value pair of $\delta$, $\rho$ do:

1. **Generate links matrix $\mathbf{A}$:**
   Draw $n$ links at random from the links set. Since $N$ is known, then $n=\text{round}(\delta N)$.
   Normalize each of the link’s values by dividing each link’s elements by the link’s length (i.e. the sum of all its elements).
   These $n$ links represent the columns in matrix $\mathbf{A}$, which is thus $N \times n$.
2. **Generate field matrix $\mathbf{X}$, and corresponding vector $\mathbf{x}$:**
   Generate a field whose sparse representation (i.e. $\mathbf{S} \cdot \text{CR}(\mathbf{X})$) has $k$ non-zeros.
   Since $n$ is known, $k=\text{round}(\rho n)$.
   $\mathbf{X}$ is $\sqrt{N} \times \sqrt{N}$.
   $\mathbf{x} = \text{CR}(\mathbf{X})$ is the column representation of $\mathbf{X}$.
   $\mathbf{x}$ is $N \times 1$.
   This generation algorithm is detailed in section 5.2.2.
3- **Calculate measurements vector \( y^n \):**

Simply calculate \( y^n = A^T x \).

\( y^n \) is \( N \times 1 \).

4- **Solve \( l_1 \) minimization:**

Reconstructing \( x' \), given \( y^n \), \( A \), and sparsifying matrix \( S \).

Solve the \( P_1 \):

\[
\min_{x_s} \| x_s \|_1
\]

Subject to: \( y^n = M x_s \)

Having \( M \equiv A^T S^{-1} \).

Remark: The solution to this step is done with the open-source CVX package (http://cvxr.com/cvx/).

5- **Derive the reconstructed rain-field \( x \):**

\[
x = S^{-1} x_s
\]

6- **Note the relative error:**

\[
E_{\text{non-sparse,rel}} = \frac{\| x - x' \|_2^2}{\| x \|_2^2}
\]

\( E_{\text{non-sparse,rel}} \) is a function of \( \delta, \rho \), and the Monte-Carlo index. Thus it is a 3-dimensional array of relative errors.

End.

End.

7- **Calculate \( E_{\text{critical}} \):**

(see 5.3.1 below).

8- **Aggregate \( E_{\text{non-sparse,rel}} \) and derive the phase transition diagram:**

\( E_{\text{non-sparse,rel}} \) is three-dimensional, to be aggregated over one dimension, the Monte-Carlo dimension.

For every point in the \((\delta, \rho)\) plane, calculate the proportion of relative square-error values that are smaller than \( E_{\text{critical}} \) (each of these values was created by a different Monte-Carlo iteration). This count is the empirical estimation of the probability of good reconstruction, meaning, the probability to have a reconstruction error below \( E_{\text{critical}} \).

Now every point in the \((\delta, \rho)\) plane has one value assigned to it, the probability of reconstructing the rainfall map with error smaller than \( E_{\text{critical}} \), this is the phase-transition diagram.

Recall that based on the fact that the Haar transform is unitary, as shown in chapter 3, this phase-diagram applies not just to the sparse field’s error, but to the original field, \( X \), as well.

This simulation mimics Donoho and Tanner’s algorithm for deriving the phase-transition diagram. As a validation of this simulation, it was applied to the same scenario that they did in their paper. A Gaussian matrix \( M \) was generated, having IID entries with mean 0 and STD 1, and so was a field \( X_s \) with non-zero entries distributed uniformly in \([-1,1]\). Figure 5.3 presents how this simulation corresponds to the analytical results of Donoho and Tanner.
Figure 5.3 - Validating that the main simulation used in this work complies with Donoho and Tanner’s. The black and white surface represents the probability of good reconstruction and generated using results from this thesis (here “good” refers to reconstruction error below $10^{-12}$, as suiting a Gaussian matrix). The blue curve is Donoho and Tanner’s result for the transition between good and poor reconstruction. $N=128\times128$, 16 Monte-Carlo iterations were ran. As seen, the blue curve corresponds to the bound between the black and the white regions.

5.3.3 The Critical Error

This thesis’s result differs from Donoho & Tanner’s by one critical detail, the value of $E_{critical}$. In their research, Donoho and Tanner choose $E_{critical}$ to take a value that would correspond to ‘exact reconstruction’. Donoho and Tanner choose to define ‘exact reconstruction’ as,

$$\frac{\|x - x’\|_2^2}{\|x\|_2^2} < E_{critical} \equiv 10^{-12}.$$

In our case, with precipitation monitoring, the benchmarks are quite different. Suggesting a relative square-error of $10^{-12}$ is still not yet within reach. The common relative square-error is in the tens of percentages (Raghavan, 2003; Peleg, 2013).

So, how would one suggest a critical error that will mark the threshold for good-reconstruction? The answer to this question stems from the results. The major product of Donoho and Tanner’s work is the transition curve of the diagram. It was derived analytically. By choosing $E_{critical} = 10^{-12}$, they found that the empirical bound stemming from their simulations closely coincides with this analytic curve. One should note that by choosing a different $E_{critical}$, the empirical bound would not coincide with the analytical curve. For $E_{critical} > 10^{-12}$, the empirical curve would be placed “higher” in the plane, and for $E_{critical} <$
$10^{-12}$, the empirical curve would be placed “lower” in the plane. Figure 5.4 illustrates the relationship between $E_{critical}$ and the bound curve.

Figure 5.4 - Generating the phase-diagram for Gaussian matrices. Here we see how the choice of critical error affects the diagram. In all the plots, the blue curve is Donoho and Tanner’s analytical bound. $N=128 \times 128$, 16 Monte-Carlo iterations were ran. Top left: the relative error plot (not the probability of error), the plotted surface is the actual relative error values, one can notice that the blue curve lay right where the error transitions from high to low. Top right: the diagram when choosing “too low” $E_{critical}$, here the surface is the probability of good reconstruction. One can notice that the transition in probability is below the blue curve. Bottom left: the diagram when choosing $E_{critical} = 10^{-12}$ just like in Donoho and Tanner’s work (same figure as 5.3). The blue curve corresponds with the transition of probability, as expected. Bottom right: the diagram when choosing “too high” $E_{critical}$. One can notice that the transition in probability is above the blue curve.

The critical relative square-error in this work can be chosen in a similar manner. Based on the empirical results for the relative square-error, $E_{critical}$ is chosen such that would allow resemblance between the consequential bound curve and the analytical curve. As Donoho and Tanner prove in their research, the resemblance is stronger as $N \to \infty$. 

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6 Results

The simulations from chapter 5 were applied to validate the main assumption, that Donoho and Tanner’s phase-transition diagram can be applied to RSL measurements via microwave-links. The results are presented here and are being translated to their meaning of rain mapping. The application of these results are farther presented in chapter 7.

6.1 The Synthetic Links

Using the algorithm for synthesizing random links presented in section 5.1.1, various links ensembles were created to allow for the exploration and analysis of various scenarios. The main variables that were explored are: the effect of the type of environment on the results (e.g. urban versus rural), and the resolution chosen for the reconstructed rain-field. It should be kept in mind that while there are two other very meaningful variables, the spatial density of links and the sparsity of the rain-field, they are embodied in the phase-transition diagram.

6.2 Results for Various Environments

Here I describe the results from the perspective of different environments such as urban, suburban, and rural. Based on the relationship between the environment type and the links distribution presented in section 4, I’ve generated simulations of scenarios and derived the appropriate results. As mentioned in the study of microwave-links, the pre-set parameter that embodies the different environments is the links’ mean length. While links density is an even clearer proxy for environment types, it isn’t a predefined parameter for the simulation, as every simulation iterates over various link densities. As presented in 5.3, the region’s area is fixed, $N$ is fixed, and $\alpha$ varies so to allow for varying $\delta$, and consequently, varying links density. Again, $N$ is fixed to be the largest value that still allows computation for the reason that Donoho and Tanner’s theory is proven analytically for $N \rightarrow \infty$. Moreover, since the links mean length is also fixed for every simulation, it is necessary to run several simulations with different mean lengths, thus creating several corresponding phase transition diagrams, so to observe the effect that the links mean length has on the empiric diagram.

In figure 6.1 are seven phase-transition curves. The first one is Donoho and Tanner’s derived curve just like in figure 2.2, and the other six are results of simulating six different links mean lengths (corresponding to six environmental densities). The first corresponds to an urban environment, having links’ lengths distributed exponentially with mean length of 1,500 meters, and the sixth corresponds to a rural environment, having links’ lengths distributed exponentially with mean length of 10,000 meters. These are visualized against Donoho and Tanner’s phase-transition curve for comparison. For all six curves, $N=128x128$ pixels, 32 Monte-Carlo iterations were ran.
Figure 6.61 - Phase transition diagrams for different environments. The thick black curve is Donoho and Tanner’s analytical asymptotic curve. Every other curve is characterized by the mean length, $\theta$, of the links that generated it. The shorter the links, the denser the environment is, as seen in the links study. 1,500m corresponds to a dense urban area, where 10,000m corresponds to a rural area.

Figure 6.1 portrays the bound curves for different scenarios, dictating whether “good” or “poor” reconstruction is expected. The attribute “good” or “poor” is with regards to a predetermined critical reconstruction error, “good” corresponding to a scenario with an error smaller than the critical error, and “poor” means the expected error is greater than the critical error. With Donoho and Tanner’s curve (black), the critical error is $10^{-12}$ (relative sum-of-squares).

The critical error assigned for each of the above six curves is a solution of a simple optimization problem. Given the surface of relative errors over the $(\delta, \rho)$ plane, $E_{relative}(\delta, \rho)$, a critical threshold must be chosen such that the corresponding contour of the relative error will have a minimum distance to the analytical curve (black curve in figure 6.1). Equation (6.2.1) portrays the optimization:

$$E_{critical} = \arg\min_{E} \left\{ \sum_{\delta} \left( \rho_{analytic}(\delta) - \rho_{E}(\delta) \right)^2 \right\},$$

(6.2.1)

Where $\rho_{E}(\delta)$ is the contour of $E_{relative}(\delta, \rho)$ when constraining it to $E_{relative}(\delta, \rho) = E_{critical}$. $\rho_{analytic}(\delta)$ is the analytical curve developed by Donoho and Tanner.

With the six links schemes that correspond to six environments, the threshold for the relative error chosen to represent the critical error is 0.1-0.13 (as mentioned in the figure’s legend).

One should appreciate the relations between the links’ diagrams to Donoho and Tanner’s while keeping in mind that the black curve is derived analytically and is considered appropriate asymptotically for $N \rightarrow \infty$. In our six links cases, $N=128^2$. 60
It is suggested in this thesis that the resemblance between the six curves means that they should all be considered one and the same (which is an approximation). Thus, when one wants to evaluate the expected ability to reconstruct a rain-field in a region of homogeneous environment (e.g., a suburb), they don’t need to look at a different curve based on the environment type. The one curve which approximately captures all six scenarios above will provide the same answer for all environments, thus the only necessary variables are \( n \), \( N \), and \( k \).

### 6.2.1 Results for Various Resolutions

Since Donoho and Tanner’s major contribution is asymptotic, it makes it hard to suggest validation for cases where \( N \) is finite. The validation is empirical, so it only comprehends computations that are feasible. The above results correspond to \( N = 128^2 \) and took a very extensive amount of time to generate. A greater \( N \) was not practically feasible given the computational resources.

Thus, one would wonder how to evaluate the resemblance between the above links’ results to Donoho and Tanner’s.

In order to provide an asymptotic insight to the convergence of the links’ phase-transition diagrams, several link diagrams were generated, all applying to the same physical scenario of links’ mean length 3600 meters, but each diagram was preset with a different \( N \). 32 Monte-Carlo iterations were applied. Figure 6.2 portrays the asymptotic effect \( N \) has on the convergence.

![Figure 6.2 - Experimenting with asymptotic convergence of the empirical microwave-links. Five values of \( N \) were experimented with, showing how these values effect the curve and the critical error associated with it.](image)

As one may notice, as \( N \) grows larges, the curves are drawn closer to the analytic curve. A more critical look hints that for \( N \) larger than portrayed here they would probably not completely
coincide with the analytical curve. Also noticeable is the convergence of the threshold of the critical error to 0.1.

6.3 Validating the Phase Transition Results With Regards to Theory

6.3.1 Validation through Simulation

The correctness of the phase-transition diagram as an estimation of expected precision is of interest. As a first and simple step, simulated scenarios were applied. New scenarios were generated to show that they yield the same precision as expected. A way to perceive this is to simply look at individual diagrams in independent Monte-Carlo iterations before being aggregated to yield the final empirical phase-transition diagram. Figure 6.3 exemplifies how independent iterations suit the final phase-transition diagram. The simulation that generated these iterations had the following values: region-area=10x10 [Km$^2$] (as in all prior calculations), $N=64x64$, mean-length=3,600 meters, 16 Monte-Carlo iterations, Haar is the sparsifying transform.

![Figure 6.3](image)

Figure 6.3 - Validating single Monte-Carlo iterations with regards to the final averaged result. Every plot is of the relative error of reconstruction (not of the probability of error). This figure shows something that isn't visible when only looking at the final resulting phase-transition diagram. It can be seen that the independent Monte-Carlo iterations roughly agree with each other and with the final result.

6.3.2 Illustration by Semi-Simulation

Here a set of true links is applied. The fields to be sampled and estimated are still synthetic as in the simulation, but the links are taken from a true set. The links here are taken from the city of Tel-Aviv, Israel. Figure 6.4 shows the region.
Figure 6.4 - A region of interest is enclosed in the black frame. This region is
over Tel-Aviv, Israel. It holds 264 links. Their density is 3.1 [links/Km$^2$], as the
region is 85.18 [Km$^2$]. It is centered over latitude:32.0545, longitude:34.8249

Figure 6.5 presents several scenarios and results. Since the point of this example is to show
how different points in the phase transition diagram correspond to different scenarios, the
resolution $N$ was limited due to the number of links. In order to maintain $\delta=0.8$, since there are
only 264 available links, $N$ is limited to 264/0.8=330 pixels. So, $N=18x18=324$ pixels. The
sparsifying transform is Haar. The links’ mean length is dictated by the given set to be 1480
meters. Each scenario in figure 6.5 is a snap shot of one iteration.
Figure 6.5 - Illustrating the relation between a scenario and its appropriate location on the diagram. Every row belongs to a different scenario. The left image in every row is the links spatial density, the middle is the original field, and the right image is the reconstructed field. a – (δ, ρ) = (0.3, 0.9), error=0.74, b – (δ, ρ) = (0.8, 0.9), error=0.57, c – (δ, ρ) = (0.8, 0.1), error=0, d – (δ, ρ) = (0.3, 0.1), error=0.35. In d the error is greater than expected. It comes to show how for a finite and small N the diagram isn’t guaranteed to capture the true error.

To put the four scenarios from figure 6.5 in context, scenarios a and b correspond to fields that are not sparse enough for reconstruction given n and N. The diagram, by knowing n, N, and k, expects poor reconstruction. In practice, the reconstruction does yield a relative error higher than the critical error of 0.1, as expected. Scenarios c and d correspond to fields that are sparse enough for reconstruction given n and N. In scenario c the reconstruction is good, as expected. Not such is the case for scenario d.

Scenario d, where (δ, ρ) = (0.3,0.1), yields a relative error of 0.35. If one were to apply the phase transition diagram for the n, N, and k of scenario d they would expect good reconstruction, i.e. a relative error below 0.1. That isn’t the case and there is an explanation. As stressed, the phase transition diagram corresponds to “large” N’s. The smaller N is, the less the phase transition diagram is expected to describe the expected reconstruction properly. Here N=18x18. One may yield a larger N by allowing for more links, e.g. by taking a larger region.
7 Discussion, Applications, Future Research, and Conclusions

7.1 The Effect of the Specific Arrangement of Links

An interesting issue to discuss is how the arrangement of the links affects the results of this work.

From basic linear algebra we know that when judging a linear system of equations, it is not enough to know the number of rows of the observations matrix, it is the rank of the observations matrix that matters. A question may be asked, is there a chance that some set of $n$ uniformly distributed links will be arranged in such a redundant manner so that the phase-diagram will predict a higher success probability than it really should? In different words, is the fact that the phase-diagram judges a set by $n$ and not by the rank of matrix $A$ compromises its integrity? Below we’ll see that while this is an interesting theoretical question, in practice one may judge $n$ to be equivalent to the rank of the matrix for the sake of calculating the phase transition diagram.

In order to answer this question the following experiment was conducted. The phase-diagram was generated twice for the same simulation. The first diagram was derived like before. The other was derived slightly differently, replacing $n$ with the rank of the matrix. Meaning, if the matrix were of full rank, then the two would yield identical results. But, if the rank were lower than $n$, then the second phase-diagram would tell a lower probabilities for reconstruction. The intuition here is driven by the fact that it is the rank that measure how informative an observations matrix is. For instance, if for some case we have $n = 100$ pointing to a probability is 0.8 and the rank is 70, then when looking at the rank instead of $n$, the probability may drop to 0.6.

Figure 7.1 shows the two phase-diagrams. Both are of $N=128\times128$ pixels, 16 Monte-Carlo iterations were used. One can see that the diagrams are practically the same.

Figure 7.1 - The difference between $n$ and the rank of the matrix, labeled $n_{\text{critical}}$. The color of the surface portrays the probability of good reconstruction. The current resolution is $N = 128^2$. The two are almost identical for every $n$. 
So, while it is the rank of the matrix that is most indicative of how informative it is, one may skip the trouble of calculating it and simply consider \( n \).

It should be stressed that the above analysis is important for when one is looking to use the diagram without performing analyses and calculations of any kind. Meaning, when one knows \( n \), \( N \), and \( k \), and wishes to see what the phase transition diagram dictates. When one does look to perform analyses and calculations, then by calculating the rank of \( \mathbf{A} \) and comparing it to the number of its columns, \( n \), one will find whether there is a difference.

### 7.2 Optimal Positioning of a Rain Gauge

A particular application of this thesis that may be of interest is the positioning of a single or several rain-gauges anywhere in the region of interest. The optimal positioning of rain-gauges had drawn studies that considered various aspects such as precipitation data and geometrical locations (Bastin et al., 1984). Here I simply present the insight that may be achieved by looking at the phase-transition diagram. This insight can complement other studies. Let us notice that a rain-gauge is a point sampling of the rain-field. It captures a single pixel (which is a single “unknown” in our linear system of equations), and autonomously provides the pixel’s value with full certainty (ignoring possible measurement errors). Nevertheless, it of course should not stand on its own. It should be integrated in some sort with the full measurement set, as it complements it and adds more information. From the perspective of the mathematical modeling, the rain-gauge is like a link with length of one pixel, fully captured by that single pixel. When added to the system, one should keep in mind that a link samples the attenuation field, while a rain-gauge samples the rain-field. One of the two should be translated to the other by the power law (see equation (1.3.2)).

When looking for the optimal placement of a rain gauge, uniformity should be maintained so to allow for optimal coverage of each region. If we were to relate spatial uniformity with minimal “bald” spots, i.e. the least amount of “bald” spots and the smallest “bald” area, then one may find the optimal location of a new rain-gauge by placing it in the center of the largest “bald” spot (figure 7.2).

![Figure 7.2](image.png)

**Figure 7.2** - An illustration of the optimal positioning of a rain-gauge so to minimize the biggest “bald” spot in the region. The X marks the rain-gauge.
7.3 How to Apply the Phase Diagram in Practice

When applying the results of this work, one should have available $\delta$ and $\rho$, i.e., $N$, $n$, and $k$.

$N$ is the wished resolution of the reconstructed rain-field, i.e. the number of pixels that the field is divided to. It translates to the spatial partition of the region of interest. For instance, if the region of interest is a square 10x10 [Km$^2$], then choosing $N=256$ will yield a resolution of 0.3906 [Km/pixel].

$n$ is the number of available links, i.e. the number of observations. For instance, if the region of interest is a square 10x10 [Km$^2$], and moreover, it is of a dense urban area, $n$ may be 300 (corresponding to a density of 3 [links/Km]).

Finally $k$ is the number of non-zeros pixels that the field is assumed to have when divided to $N$ pixels. Meaning, $0 \leq k \leq N$. For instance, if $N$ is set to 256 pixels, $k$ may take on any value up to 256 (corresponding to non-sparsity).

$n$ is usually simple to yield, as $n$ is dictated by the scenario. $N$ may be optimized to suit application needs. $k$, on the other hand, is the least trivial and requires prior study such as presented in section 4.3.

7.3.1 Calculating Appropriate Sparsity

Another way one may apply the diagram is to examine a certain setup. Meaning, one can have $n$ and $N$ based on demands or constraints, and the goal is to understand what rain-field this setup may serve.

As an example, let’s revisit the work done by SMHI (https://www.ericsson.com/mobility-report/microweather-unlocking-potential). They present a selected region and also the links distribution (figure 7.3).

![Figure 7.3 - The scenario of SMHI’s experiment, “The Gothenburg Pilot”. On the left is the region Gothenburg with the distribution of microwave-links. On the right is the reconstructed rain-field via the microwave-links’ measurements.](image-url)
From that data we can take the number of links, \( n = 364 \), and resolution \( N = 41 \times 41 = 1681 \). Note that in the data there are actually 364 “double-links”, hops, which are links that preform transmission and reception to both directions in different frequencies. Since the mathematical model in this thesis assumes all links apply similar frequencies, and moreover, the frequencies correspond to \( b = 1 \) (equation (1.3.3)), I’ll only regard it as if only 364 links provide information.

So, \( \delta = \frac{n}{N} = 0.217 \). Figure 7.4 portrays the relevant \( \rho \) for this \( \delta \). According to the diagram, the critical \( \rho \) is,

\[
\rho_{\text{critical}} = 0.25,
\]

and thus the critical sparsity is,

\[
s_{\text{critical}} = \rho_{\text{critical}} \cdot \frac{n}{N} = 0.25 \cdot 0.217 = 0.054 = 5\%.
\]

So, in order to expect this system to allow for good reconstruction (i.e. a reconstruction error below 10%) given this resolution, the rain-fields must have a sparsity rate of 5% when presented in Haar domain. Based on the image examples of the differentiation transform (appendix B) and the Haar transform (chapter 3), a 5% sparsity rate is definitely feasible, but farther study must be done to assure that. The sparsity study of radar images in section 4.3 showed that very few of the images present sparsity of 5% or less when using Haar as a sparsifying transform.

![Figure 7.4](image)

Figure 7.4 - Applying the phase-transition diagram to finding critical \( \rho \).

Given \( \delta = 0.217 \) we can trace the corresponding \( \rho_{\text{critical}} = 0.25 \).

### 7.3.2 Calculating Optimal Resolution

We would like to achieve reconstruction of the rain-field. Since this is a function of the wished resolution, we would like to simply find the best resolution that would allow good reconstruction. Here we assume that we are given a specific amount of links measurements, \( n \). This assumption is the common assumption for the time this work is written, as the common approach is to take advantage of existing cellular infrastructure without the ability to affect it and increase the number of microwave-links (i.e. \( n \)). Also given is the region of interest. It is of a homogeneous environment type (e.g. urban). Based on the argument in section 6.2, it does not matter which
environment type is presented in the region as it is approximated that the phase-transition diagram roughly applies to all types.

The approach here is to evaluate the probability of good reconstruction for every \( N \) so to eventually choose the best \( N \).

**Pre-set variables:**

1. \( n \) is given:
   The number of microwave-links that measure the field. Note that their topology isn’t being evaluated, only their count. This is a major contribution that this entire work and the phase transition diagram present. They provide a generic evaluation without having to evaluate the specific topology for every specific microwave-links network.

2. A set of historical rain-field images that apply to the relevant region of interest:
   The images must be available in a “high enough” resolution, as they are used to exemplify what resolutions are relevant. Ideally, one would want to have the true continuous images, which isn’t feasible for a digital machine. A practical source may be a radar or satellite image set.

3. A list of potential resolutions \( \{N_i\} \) to examine:
   The smallest one may be as small as 1 pixel (for when looking to derive one global rain-value for the entire region), and the largest one cannot be greater than the resolution of the given rain-fields, as that is the most informative data we have. All \( N \)'s are integers that comply with the region’s proportions, i.e. if the region is square then all \( N \)'s are squared integers.

**The resolution optimization algorithm:**

For every \( N \) in \( \{N_i\} \), do:

1. **Down-sample all the given images to resolution \( N \)**
2. **For every image, derive the \( \rho \) level:**
   Count all non-zero pixels in an image and divide by \( n \).
3. **Derive a probability density function for \( \rho \):**
   There are various methods, the simplest one is to calculate a histogram.
4. **Calculate \( P(\text{reconstruction}|N, n) \):**
   Given \( \delta \) (e.g. \( n/N \)), and using the phase transition diagram, estimate the probability for the sparsity to allow good reconstruction.
   Meaning, calculate \( P(\rho < \rho_{critical}) \).

The result is a function of \( N \), \( P(\text{reconstruction}|N, n) \) for all \( N \) in \( \{N_i\} \). \( N \) may be then chosen based on a probability that is satisfactory to the user.

It should be noted that ironically, one may actually prefer microwave-links over weather radar with the intention to yield better resolution. So in that case, it wouldn’t make sense to limit the resolution exploration to the radar’s resolution. In that case, a different source of rain-field images is necessary. One approach may be to synthesize rain-fields based on prior research, and generate them with a resolution as high as wished to examine. Figure 7.5 shows an example to the application.
Figure 7.5 - Example of a single iteration of resolution optimization. The number of links is given and usually can’t be controlled. A set of characteristic images are being down-sampled to the resolution \(N\) of the current iteration (step 1). \(\rho\) is calculated for each image (step 2). Notice that \(\rho\) gets a maximum value of \(N/n\). Having all \(\rho\)'s, a probability density of \(\rho\) is estimated (step 3). Last, given \(\delta = n/N\), \(\rho_{\text{critical}}\) is derived from the diagram, and the probability of \(\rho < \rho_{\text{threshold}}\) is calculated. It is the probability of reconstruction (step 4).
As a final demonstration of the method for resolution evaluation, let’s take another look at SMHI’s data. Section 4.3 presented SMHI’s radar images and the sparsity that they dictate. The region was in south Sweden. Here the region is narrowed down to the Gothenburg city area.

As explained in section 7.3.1, $n=364$ is the number of links. The number of pixels, $N$, available in the images for the relevant region of Gothenburg is 32X32. The area is 64x64 Km$^2$, thus corresponding to the resolution described in section 4.3, 4 [Km$^2$/pixel] (figure 7.6). Having that resolution is 32X32 pixels, there is little room to explore. Below I evaluate whether in fact the current resolution is reconstructable given the number of links.

![Figure 7.6 - The Gothenburg region (small square). 288 images were studied for sparsity.](image)

Evaluating the ability to reconstruct:

$N = 32 \times 32$,

$\delta = \frac{364}{32^2} = 0.356$,

$\rho_{\text{critical}} = 0.309$,

The $\rho$ histogram of the 288 radar images was generated. The probability of a field being sparse enough is:

$P(\rho \leq \rho_{\text{critical}}) = 0.323$. 

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So, with the given amount of links, 364, the probability of achieving good reconstruction is low. One should note that each link is a double-link, allowing cellular RSL measurements in both directions. If one were to be able to achieve a reconstruction from such double links as if they were separate independent links, then we would actually count \( n = 728 \) links. In which case:

\[
N = 32 \times 32,
\]

\[
\delta = \frac{728}{32^2} = 0.712,
\]

\[
\rho_{\text{threshold}} = 0.5,
\]

Using the same \( \rho \) histogram, the probability of having a rain-field comply to the sparsity is:

\[
P(\rho \leq \rho_{\text{threshold}}) = 0.365.
\]

Still, not promising.

The next step would be to compromise on a lower resolution. For instance, if every four pixels were grouped to one, we would get a resolution of 16X16. This would mean that there are less pixels then links. Thus, depending on the linear dependencies of the rows in the links matrix, \( A \), reconstruction would be feasible.

### 7.5 The Limitation of this Research

It is important to stress the limitations of the results derived in this thesis. The major product of this thesis is the phase transition diagram’s application to the reconstruction of rain-fields. There are two foundations that this result relies upon: the use of the \( P_1 \) algorithm (equation (2.2.2)) to solve the mathematical problem (equation (3.4.1)), and the choice of the Haar transform as a sparsifying transform. The former was chosen to correspond with and build on Donoho and Tanner’s research (Donoho and Tanner, 2009), and the latter was chosen after experimenting with various unitary transformations and choosing the one that allows for the best reconstruction results (which corresponds directly with being the best sparsifying transform out of those examined). If one were to apply a different reconstruction algorithm (e.g. by hypothetically applying the \( P_0 \) algorithm (equation (2.2.1))), or choose a different sparsifying transform, then different, and perhaps better reconstruction results would be derived.

Choosing different solution algorithms and/or different sparsifying transforms may affect the results of this thesis in two possible manners:

1- The phase transition diagram may still apply just like it does in this thesis, but the critical relative error will change. For instance, a different unitary transform may sparsify the rain-fields much more effectively such that the critical relative error won’t be 0.1 but 0.01. In this situation, since the transform is unitary, Donoho and Tanner’s analytical phase transition curve will still describe the transition from good reconstruction to poor (see section 3.5 for proof).

2- The phase transition diagram may no longer apply. One may suggest a different reconstruction solution such that the relative error will behave differently, not corresponding deterministically with the phase transition diagram. See appendix B where the differentiation transform is presented. The error doesn’t seem to correspond to
Donoho and Tanner’s phase transition diagram. Moreover, this method often yielded reconstruction errors much smaller than Haar did.

The choice of reconstruction methods ($P_1$, algorithm, and Haar) in this thesis was to enable criteria of reconstruction. It is possible that different methods would allow better field reconstruction results. However, right now there is not better method for deriving reconstruction expectation than this thesis does. Moreover, there is no other work suggesting results of this nature.

7.6 Future Research

The work presented here can be leveraged in a few directions. A few ideas that would complement this research are:

- **Sparsifying transform.**
  There is room to farther explore sparsifying transforms that may represent the typical rain-fields in a sparser way, thus allowing for a better reconstruction.

- **Rain-fields sparsity.**
  Being able to characterize the sparsity of rain-fields is essential for applying results of this work. For this sake I explored Gothenburg’s rain-fields, but there is much room to get a clearer understanding. Especially, since the choice of resolution $N$ affects the sparsity, it would be valuable to suggest a formula relating the fields’ sparsity to the chosen $N$.

- **Integrating other weather instruments.**
  How would integrating radar, gauges, and other available instruments’ observations affect the performance? This ensemble may take on two approaches. The one is to have all instruments yield their estimation of the rain-field, and then to aggregate them to one field. The other would be to suggest an integrated mathematical model, perhaps similar to the one presented in this thesis, and then to solve that one model.

- **Modeling observations noise.**
  The solution to the system of linear-equations that is presented here is the $P_1$ problem. It minimizes $||x||_1$, while constraining $y^n = Mx$. On the other hand, there may be benefit to allow for a noise factor $\varepsilon$, by relaxing the constraint to be $||y^n - Mx||_p \leq \varepsilon$.

7.7 Conclusion

As the problem statement described, this work provides three contributions. The main contribution is to the field of precipitation monitoring, by suggesting a mathematical criteria for feasibility of rain-field reconstruction via microwave-links measurement. The other two are consequential contributions, developed to serve the main one. The second of the three is the expansion of results from Compressed-Sensing. The adaptation here was to expend available criteria to the measurements of microwave-links. The last contribution of the three is the statistical study and modeling of microwave-links.
7.7.1 Rainfall Monitoring

This work presented a novel answer to the question of when a rainfall-map can be reconstructed. The solution allows for a quick and simple insight as to whether a given scenario allows for reconstruction. Being practical, the work here also specifies the reconstruction algorithm. Moreover, the solution is expended to allow one to understand what may be adjusted so to allow reconstruction, namely, trading-off spatial resolution with feasible reconstruction.

7.7.2 A Broader Scientific Contribution: Images Sampled by Random Line Projections

While the use of Compressed-Sensing came to serve this thesis’s main goal for rain-field reconstruction, it actually tends to a broader mathematical question. If an image is to be sampled by collecting projections of it along random straight lines, could it be reconstructed? As Sendik tackled the rainfall reconstruction problem in 2012, he suggested a different approach to the solution. Nevertheless, he too realized that by providing a solution to our rainfall reconstruction problem, he consequently provided a solution to the broader problem of general image sampling via random projections (Sendik and Messer, 2012). And so, as my approach is quite different then Sendik’s, I too present a novel solution to telling whether an image can be reconstructed from random line projections.

7.7.3 Microwave-Links Modeling

This work applies statistical properties of microwave-links from a spatial perspective. As this modeling was vital for the suggested approach, there was little-to-no information about it in the literature. The decision to explore and model random microwave-links as a part of this thesis was thus inevitable. Its results were encouraging. The properties of the locations, lengths, and orientations of the links were clear, logical, and corresponded with the little information that I did manage to find (e.g. a study on cellular base stations). This statistical exploration and modeling can be used to serve farther studies that involve microwave-links, and in particular, rainfall monitoring via microwave-links.
8 References


9 Appendices

9.1 A – 2-D Haar Transformation in Column Representation

In this section I will formulate a method to perform the 2-D Haar transform that suits the operational constraint of our problem.

In our problem we are given a vector \( y^N \) such that \( y^N = A^T x \).

We are interested in applying a mathematical manipulation that will yield \( y^N = M x_s \), such that \( x_s \) is a column-representation of the 2-D field \( Haar[X] \). \( Haar[X] \) is the 2-D Haar transform of the field \( X \).

More formally:

*Find a non-singular \( S \in \mathbb{R}^{N \times N} \), such that \( S \cdot CR[X] = CR[Haar[X]] \),

where \( CR[\cdot] : \mathbb{R}^{ij} \rightarrow \mathbb{R}^{ij \times 1} \), performs the column-representation rearrangement of a matrix.*

Given such \( S \), we can write \( y^N = A^T x = A^T S^{-1} S x = M x_s, \quad M = A^T S^{-1} \).

Notice that \( CR[aA + bB] = a CR[A] + b CR[B] \) is a linear operator. It is easy to verify as the operator simply shifts the components.

It is also known that that Haar transform is also a linear operator.

Let’s represent any arbitrary \( X \in \mathbb{R}^{N \times \sqrt{N}} \) as a summation over all its dimensions:

\[
X = \sum_{i=1}^{\sqrt{N}} \sum_{j=1}^{\sqrt{N}} [X]_{i,j} e_{i,j},
\]

Where matrices \( e_{i,j} \in \mathbb{R}^{N \times \sqrt{N}} \) has “1” in the \( i \)-th row and the \( j \)-th column, and “0” anywhere else.

So,

\[
CR[Haar[X]] = CR \left( \sum_{i=1}^{\sqrt{N}} \sum_{j=1}^{\sqrt{N}} [X]_{i,j} e_{i,j} \right) = CR \left( \sum_{i=1}^{\sqrt{N}} \sum_{j=1}^{\sqrt{N}} [X]_{i,j} Haar\{e_{i,j}\} \right) =
\]

\[
\sum_{i=1}^{\sqrt{N}} \sum_{j=1}^{\sqrt{N}} [X]_{i,j} CR\{Haar\{e_{i,j}\}\} = \left\{ \begin{array}{ll}
Let's set a new index:
\end{array} \right.
\]

\[
\text{the column-representation rearrangement.}
\]

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So, we found our matrix:

\[
\text{Here has a "1" in the } j\text{-th spot where } p = (j - 1)\sqrt{N} + i, \text{ and "0" elsewhere.}
\]

This matrix is invertible, as \{e_p\} is an orthogonal set, and the Haar transform preserves orthogonality.

9.1.1 S is Unitary

As \(S\)'s columns are orthogonal, we get \(S^T s_j = \delta_{i,j}\), \(\delta_{i,j}\) being Kronecker's delta.

So,

\[
S^T S = SS^T = \begin{bmatrix}
\delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,N} \\
\delta_{2,1} & \delta_{2,2} & \cdots & \delta_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{N,1} & \delta_{N,2} & \cdots & \delta_{N,N}
\end{bmatrix} \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]
9.2 B - Experimenting with Another Transformation – Differentiating to Allow Sparsity

This part is meant to present an experimentation I had with another transform before realizing that the transform must be isometric. It is the failure of this transform that drove me to seek mathematical conditions that the transform must possess. Here I present the process I went through and show an example of how this specific transformation failed.

Here I attempted to sparsify field images by differentiating them. See figure 9.1 for illustrations.

This approach was in fact the original sparsifying approach in this thesis. As it happened, there is a major flaw with this choice for a sparsifying transform. It stems from the fact that it does not possess an important quality, isometric. As explained, Haar transform’s quality of norm preserving dictates that by knowing the relative error for the sparse field, one also knows the relative error of the rain-field as derived after applying the inverse transform. It will be shown that the differentiation approach offers no way to relate the given relative error of the sparse field to the eventual relative error of the rain-field. It is possible to have one scenario where both relative errors are 0.1, and a different scenario where the relative error for the sparse field of 0.1 but the relative error of the rain-field is 0.5. The reason is that with this transform, it is not only the magnitude of the error that matters, but also the location of the error pixels.

Let $S$ now be the matrix that differentiates $x$ in both the $x$ and the $y$ directions. $S$ is $MxN$ and is designed to be applied to the column representation $x$ of $X$, such that the outcome is identical to the column representation of $X$ after being differentiated in the horizontal and vertical axis. This is formally explained and also $S$’s structure is detailed in Appendix C.
Figure 9.1: Portraying the sparsifying effect of the differentiation transform on tentative attenuation-fields. Every row has a tentative attenuation-field on the left and its transform on the right.)
9.2.1 The Reconstruction Stages

Once we derived the equation form \(-y^n = Mx_s\), then the problem is identical to the Haar case. It is the same \(P_1\) problem in (2.2.2).

9.2.2 The Reconstruction Quality of the Original Non-Sparse Field: A Problem

Once the sparsifying transform is applied and the mathematical model, \(y^n = Mx_s\) (\(x_s\) being sparse), then the phase-diagram may applies (this thesis revolves around that). The problem stems from the fact that this transform is not isometric, thus the phase-diagram is inapplicable to the reconstruction of the original field, \(x\).

Prior to deriving the need for isometry, as I was exploring the simulations I noticed that while the relative error of the sparse field, \(x_s\), presented a phase-transition curve very similar to Donoho and Tanner’s analytical curve, the relative error of the attenuation-field, \(x\), did not present a very clear and sharp transition telling between good reconstruction and poor reconstruction. See the two relative errors in figure 9.2.

![Figure 9.2: Resulting relative errors for when sparsifying with differentiation. On the left is the relative error of the sparse (differentiated) field, and on the right is the attenuation-field’s. As one may notice, the left figure portrays a sharp phase-transition that is similar to Donoho and Tanner’s, the right figure doesn’t. One should not judge the phase transition on the left as if it should resemble Donoho and Tanner’s since in this case \(N\) was chosen to be \(32^2\), while in other simulation in this report it is \(128^2\).](image)

In order to portray how this transform fails, I will simply present an example with two sparse fields with identical relative error and show how they end up with two very different attenuation-field relative errors. By being so, it completely eliminates the contribution that the phase-transition diagram has to this application, as its power comes from the ability to predetermine the relative error for the sparse field. If the relative error of the sparse field isn’t deterministically and reliably predictive of the attenuation-field error, then it is of little use for this application.
Consider the following attenuation-field (or rain-fields with the proper scaling) to be reconstructed after applying our technique:

\[
X_1 = \begin{bmatrix} 1 & 7 & 7 \\ 1 & 7 & 7 \\ 1 & 1 & 1 \end{bmatrix},
\]

and its sparse representation to be reconstructed:

\[
X_{1,s} = D_yX_1D_x = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 7 & 7 \\ 1 & 7 & 7 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & -6 & 0 \end{bmatrix}.
\]

Assume the reconstructed sparse field isn’t perfect and it is:

\[
X'_{1,s} = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & -6 & 0 \end{bmatrix}.
\]

Thus the error is:

\[
E_{1,s} = X_{1,s} - X'_{1,s} = \begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & -6 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & -6 & 0 \end{bmatrix}.
\]

The relative error of the sparse field is:

\[
E_{1,s-relative} = \frac{\text{SoS}(E_{1,s})}{\text{SoS}(X_{1,s})} = \frac{1}{73} = 0.0137,
\]

where SoS is the sum of squares of the elements.

Now let’s consider a new scenario with a different original attenuation-field:

\[
X_2 = \begin{bmatrix} 3 & 1 & 1 \\ 3 & 3 & 1 \\ 3 & 3 & 3 \end{bmatrix},
\]

with a sparse representation to be reconstructed:

\[
X_{2,s} = D_yX_2D_x = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2.5852 \end{bmatrix}.
\]

Assume the reconstructed sparse field is:

\[
X'_{2,s} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2.5852 \end{bmatrix}.
\]

Thus the error is:

\[
E_{2,s} = X_{2,s} - X'_{2,s} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2.5852 \end{bmatrix} - \begin{bmatrix} 3 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2.5852 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.5852 \end{bmatrix}.
\]
And the relative error of the sparse field is:

\[ E_{2,s,relative} = \frac{SoS(E_{2,s})}{SoS(X_{2,s})} = \frac{0.3425}{25} = 0.0137. \]

So, we have a situation where both relative errors of the sparse fields are the same.

But let’s see what the attenuation-field errors are:

\[ \mathbf{X}'_1 = \mathbf{D}^{-1}_y \mathbf{X}'_{1,s} \mathbf{D}^{-1}_x = \begin{bmatrix} 1 & 0 & 0 & 0 & 6 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & -6 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 6 \\ 6 & 6 & 6 \\ 0 & 0 & 0 \end{bmatrix}. \]

Thus the error is:

\[ \mathbf{E}_1 = \mathbf{X}_1 - \mathbf{X}'_1 = \begin{bmatrix} 1 & 7 & 7 \\ 1 & 7 & 7 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 6 & 6 \\ 0 & 6 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \]

And the relative error of the attenuation-field is:

\[ E_{1,relative} = \frac{SoS(E_1)}{SoS(X_1)} = \frac{9}{201} = 0.0448. \]

And for the second case:

\[ \mathbf{X}'_2 = \mathbf{D}^{-1}_y \mathbf{X}'_{2,s} \mathbf{D}^{-1}_x = \begin{bmatrix} 1 & 0 & 0 & 3 & -2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 2 & -2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 2.5852 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 3 & 3 & 1 \\ 3 & 3 & 3.5852 \end{bmatrix}. \]

Thus the error is:

\[ \mathbf{E}_2 = \mathbf{X}_2 - \mathbf{X}'_2 = \begin{bmatrix} 3 & 1 & 1 \\ 3 & 3 & 1 \\ 3 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 1 \\ 3 & 3 & 1 \\ 3 & 3 & 3.5852 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.5852 \end{bmatrix}. \]

And the relative error of the attenuation-field is:

\[ E_{2,relative} = \frac{SoS(E_2)}{SoS(X_2)} = \frac{0.3425}{57} = 0.006. \]

To conclude, while both cases had the same relative-error of 0.0137 for the sparse fields, they actually have attenuation-field relative errors differing by about a magnitude of 10 (and the same is true for the corresponding rain-fields). The key to understanding this phenomenon is to understand that the location of the error for the first case is in the top left corner of the region, and for the second, it is in the bottom left. Since the inverse operator to the differentiation is an integration starting at the top left corner of the region, then the first relative-error is propagating to the entire region of the attenuation-field, while the second did not.

On a different note, it is noticed by the above images that this differentiation transform allows for much more sparsity. Meaning, a vector $x$ is more sparse after the differentiation transform than after the Haar transform. Moreover, experimenting with this transform has actually showed that it reconstructs the attenuation-field even better than when using Haar’s. Its downside is that
it does not allow a clear expected error like Donoho and Tanner’s diagram attempts to do. So for the description of the bound, Haar does a perfect job, but if one were to try to achieve better reconstruction results, perhaps it is worthwhile to look farther into the differentiation transform.
9.3 C – 2-D Differentiation in Column Representation

In this section I will formulate a method to apply a 2-D differentiation operator that suits the operational constraint of our problem.

In our problem we are given a vector $y^n$ such that $y^n = A^T x$.

We are interested in applying a mathematical manipulation that will yield $y^n = M x_s$, such that $x_s$ is a column-representation of the 2-D field $D_y \{D_x \{X\}\}$. $D_y \{D_x \{X\}\}$ is the 2 directional differentiation of the field $X$.

$$D_y \{D_x \{X\}\} = D_y \cdot \{D_x \{X\}\} = D_y \cdot X \cdot D_x =$$

$$\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & 1
\end{bmatrix} \cdot 
\begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}$$

For instance, for a 4x4 case:

$$\begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix} \cdot 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 3 & 4 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = 
\begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix} \cdot 
\begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
0 & -3 & -1 & 4
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & -2 \\
0 & 2 & 0 & -2 \\
0 & -3 & -1 & 4
\end{bmatrix}$$

Remark:

Notice that as oppose to the continuous differentiation, these discrete differentiations are invertible. These two matrixes are not singular.

Formalizing our need:

Find a non-singular $S \in \mathbb{R}^{N \times N}$,

Such that $S \cdot CR(X) = CR \{D_y \cdot X \cdot D_x\}$,

where $CR\{\} : \mathbb{R}^{i \times j} \to \mathbb{R}^{i' \times j'}$, performs the column-representation rearrangement of a matrix.

Given such $S$, we can write $y^n = A^T x = A^T S^{-1} S x = M x_s$, $M = A^T S^{-1}$.

As oppose to appendix A where I applied analytical manipulations, here we can figure out the structure of $S$ by following the interactions between the components.

Let’s first find $S_y$ that handles the case of differentiation along the horizontal dimensions:
$S_y \cdot CR\{X\} = CR\{D_y \cdot X\}$.

$D_y \cdot X = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & 1
\end{bmatrix} \begin{bmatrix}
[X]_{1,1} & [X]_{1,2} & [X]_{1,3} & \cdots & [X]_{1,\sqrt{N}} \\
[X]_{2,1} & [X]_{2,2} & [X]_{2,3} & & [X]_{2,\sqrt{N}} \\
[X]_{3,1} & [X]_{3,2} & [X]_{3,3} & & [X]_{3,\sqrt{N}} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
[X]_{\sqrt{N},1} & [X]_{\sqrt{N},2} & [X]_{\sqrt{N},3} & \cdots & [X]_{\sqrt{N},\sqrt{N}}
\end{bmatrix} = \begin{bmatrix}
[X]_{1,1} & [X]_{1,2} & [X]_{1,3} & \cdots & [X]_{1,\sqrt{N}} \\
[X]_{2,1} - [X]_{1,1} & [X]_{2,2} - [X]_{1,2} & [X]_{2,3} - [X]_{1,3} & \cdots & [X]_{2,\sqrt{N}} - [X]_{1,\sqrt{N}} \\
[X]_{3,1} - [X]_{2,1} & [X]_{3,2} - [X]_{2,2} & [X]_{3,3} - [X]_{2,3} & \cdots & [X]_{3,\sqrt{N}} - [X]_{2,\sqrt{N}} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
[X]_{\sqrt{N},1} - [X]_{\sqrt{N}-1,1} & [X]_{\sqrt{N},2} - [X]_{\sqrt{N}-1,2} & [X]_{\sqrt{N},3} - [X]_{\sqrt{N}-1,3} & \cdots & [X]_{\sqrt{N},\sqrt{N}} - [X]_{\sqrt{N}-1,\sqrt{N}}
\end{bmatrix}$

So,

$CR\{D_y \cdot X\} = \begin{bmatrix}
[X]_{1,1} \\
[X]_{2,1} - [X]_{1,1} \\
[X]_{3,1} - [X]_{2,1} \\
\vdots \\
[X]_{\sqrt{N},1} - [X]_{\sqrt{N}-1,1} \\
[X]_{1,2} \\
[X]_{2,2} - [X]_{1,2} \\
\vdots \\
[X]_{\sqrt{N},2} - [X]_{\sqrt{N}-1,2}
\end{bmatrix} = \begin{bmatrix}
D_y & 0_{\sqrt{N}\times\sqrt{N}} & \cdots & 0_{\sqrt{N}\times\sqrt{N}} \\
0_{\sqrt{N}\times\sqrt{N}} & D_y & \cdots & 0_{\sqrt{N}\times\sqrt{N}} \\
\vdots & \vdots & \ddots & \vdots \\
0_{\sqrt{N}\times\sqrt{N}} & 0_{\sqrt{N}\times\sqrt{N}} & \cdots & D_y
\end{bmatrix} \begin{bmatrix}
[X]_{1,1} \\
[X]_{2,1} \\
\vdots \\
[X]_{\sqrt{N},\sqrt{N}}
\end{bmatrix}$

$= S_y \cdot CR\{X\}$.

Here $0_{\sqrt{N}\times\sqrt{N}}$ is a matrix of zeros with dimensions $\sqrt{N} \times \sqrt{N}$.

Notice also that $S_y$ is invertible. There are two way to see this. One, it is block-diagonal, where the blocks are all invertible matrices. Two, it performs an operation that is invertible, as was mentioned above, the operation of $D_y$ is invertible.

So far we have $CR\{D_y \cdot X \cdot D_x\} = S_y \cdot CR\{X \cdot D_x\}$.

Now let's try to find:

$S_x \cdot CR\{X\} = CR\{X \cdot D_x\}$.

$X \cdot D_x = \begin{bmatrix}
[X]_{1,1} & [X]_{1,2} & [X]_{1,3} & \cdots & [X]_{1,\sqrt{N}} \\
[X]_{2,1} & [X]_{2,2} & [X]_{2,3} & & [X]_{2,\sqrt{N}} \\
[X]_{3,1} & [X]_{3,2} & [X]_{3,3} & & [X]_{3,\sqrt{N}} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
[X]_{\sqrt{N},1} & [X]_{\sqrt{N},2} & [X]_{\sqrt{N},3} & \cdots & [X]_{\sqrt{N},\sqrt{N}}
\end{bmatrix} \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$
A matrix with two diagonals: the main diagonal has 1's, and the diagonal that is \( N \) columns to the left of the main diagonal is -1's.

Here is an example:

\[
\begin{bmatrix}
[X]_{1,1} & [X]_{1,2} - [X]_{1,1} & [X]_{1,3} - [X]_{1,2} & \cdots & [X]_{1,\sqrt{\sqrt{N}}} - [X]_{1,\sqrt{N}-1} \\
[X]_{2,1} & [X]_{2,2} - [X]_{2,1} & [X]_{2,3} - [X]_{2,2} & \cdots & [X]_{2,\sqrt{\sqrt{N}}} - [X]_{2,\sqrt{N}-1} \\
[X]_{3,1} & [X]_{3,2} - [X]_{3,1} & [X]_{3,3} - [X]_{3,2} & \cdots & [X]_{3,\sqrt{\sqrt{N}}} - [X]_{3,\sqrt{N}-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
[X]_{\sqrt{\sqrt{N}},1} & [X]_{\sqrt{\sqrt{N}},2} - [X]_{\sqrt{\sqrt{N}},1} & [X]_{\sqrt{\sqrt{N}},3} - [X]_{\sqrt{\sqrt{N}},2} & \cdots & [X]_{\sqrt{\sqrt{N}},\sqrt{\sqrt{N}}} - [X]_{\sqrt{\sqrt{N}},\sqrt{\sqrt{N}-1}}
\end{bmatrix}
\]

So,

\[
CR\{X \cdot D_x\} = \begin{bmatrix}
[X]_{1,1} \\
[X]_{2,1} \\
[X]_{3,1} \\
\vdots \\
[X]_{\sqrt{\sqrt{N}},1} \\
[X]_{1,2} - [X]_{1,1} \\
[X]_{2,2} - [X]_{2,1} \\
\vdots \\
[X]_{\sqrt{\sqrt{N}},\sqrt{\sqrt{N}}} - [X]_{\sqrt{\sqrt{N}},\sqrt{\sqrt{N}-1}}
\end{bmatrix} = S_x \begin{bmatrix}
[X]_{1,1} \\
[X]_{2,1} \\
\vdots \\
[X]_{\sqrt{\sqrt{N}},\sqrt{\sqrt{N}}} - [X]_{\sqrt{\sqrt{N}},\sqrt{\sqrt{N}-1}}
\end{bmatrix} = S_x \cdot CR\{X\}.
\]

\( S_x \) is a matrix with two diagonal: the main diagonal has 1's, and the diagonal that is \( \sqrt{N} \) columns to the left of the main diagonal is -1's.

Here is an example for \( N = 9 \):

\[
CR\{X \cdot D_x\} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
[X]_{1,1} \\
[X]_{2,1} \\
[X]_{3,1} \\
[X]_{1,2} - [X]_{1,1} \\
[X]_{2,2} - [X]_{2,1} \\
[X]_{3,2} - [X]_{3,1} \\
[X]_{1,3} - [X]_{1,2} \\
[X]_{2,3} - [X]_{2,2} \\
[X]_{3,3} - [X]_{3,2}
\end{bmatrix}
\]

\( S_x \) is also invertible. Again, it represents an operation that is invertible. Also, its determinant is relatively easy to calculate.

So, we finally have,

\[
CR\{D_y \cdot X \cdot D_x\} = S_y \cdot CR\{X \cdot D_x\} = S_x \cdot S_y \cdot CR\{X\} = S \cdot CR\{X\}.
\]

\( S = S_x \cdot S_y \).
9.4 D – Regions of Israel Used for Analyses of Microwave-Links

All the links presented here are of a single cellular provider, Cellcom.

9.4.1 All of Israel

- Center of region:
  Latitude: 32.985
  Longitude: 36.65
- Area of region:
  22,770 [Km$^2$]

Cellcom links (dated 2013):

- Number of links:
  3,624
- Density:
  0.16 [link/Km$^2$]
- Mean link length:
  3.54 [Km$^2$]

See the following figure 9.3 for the region.

Figure 9.3 - All of Israel.

Figure label: All of Israel.
9.4.2 Ramat Hagolan

- Center of region:
  Latitude: 32.985
  Longitude: 36.65
- Area of region:
  1341.15 [Km$^2$]

Cellcom links (dated 2013):

- Number of links: 98
- Density: 0.07 [link/Km$^2$]
- Mean link length: 5.51 [Km$^2$]

See the following figure 9.4 for the region.

Figure 9.4 - Ramat Hagolan. The black rectangle marks the observed area.
9.4.3 Hasharon

- Center of region:
  Latitude: 32.225
  Longitude: 34.905
- Area of region:
  235.54 [Km$^2$]

Cellcom links (dated 2013):

- Number of links:
  124
- Density:
  0.53 [link/Km$^2$]
- Mean link length:
  2.5 [Km$^2$]

See the following figure 9.5 for the region.

Figure 9.5 - Hasharon. The black rectangle marks the observed area.
9.4.4 Tel-Aviv

- Center of region:
  Latitude: 32.0545
  Longitude: 34.8249

- Area of region:
  85.18 [Km²]

Cellcom links (dated 2013):

- Number of links:
  264

- Density:
  3.1 [link/Km²]

- Mean link length:
  1.48 [Km²]

See the following figure 9.6 for the region (a more detailed figure is 6.4).
Abstract

Shiite Discovery of a new method -2006 " Prof. Yosif M. et al. (2006)"

A new method for detecting air pollutants using wireless networks, with the ability to use measurement and transmission data.

This new method is revolutionary in its approach.

Current methods are limited in their spatial resolution. Since the publication, many studies have been conducted in this field, focusing on the geometrical aspects of the method, data processing, and the analysis of the data, among other topics.

The purpose of the research presented here is to establish criteria for the successful reproduction of pollution maps.

These criteria, which are used in our field, take into account the network data and the network output, and calculate the probability of successful reproduction.

In other words, given a set of cellular measurements, these criteria provide an immediate prediction of the expected pollution map.

Moreover, the development also provides guidelines for the user in designing a system, by defining how to improve the output to achieve a higher probability of success.

The methods used in this study include image processing, linear operators, and statistical modules.

The contribution of this research is threefold. First, the current research provides an answer to the question of research and thus completes another piece of the puzzle of the pollution monitoring system, which is of great importance to science and industry.

Second, it represents another contribution to the study of linear equation systems and their use in solving data measurement problems. In particular, a specific scheme is presented for sampling images that have a statistical distribution over random lines.

Third, the research includes statistical analysis of cellular links, which is then used to create a statistical model of cellular links, enabling the creation of large-scale simulations as desired.

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מיפוי גשמים תוך שימוש בחישה דחוסה: קריטריונים לשחרור המקשים之间的 שניים כדי למפות, רזולוציות המפת והדגימות ראשיות שלה.

חברון זה הוגש כעבודת גמר נוכחת התחום "ממסכן אוניברסיטי" בהנחיית שמידל אקולוסטיקית.

על - דיר

ליאור גזית

העבורה央企ית_Finalטשת של ממסכן אוניברסיטי, עםית

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